Many Sequential Iterative Algorithms Can Be Parallel and (Nearly) Work-efficient

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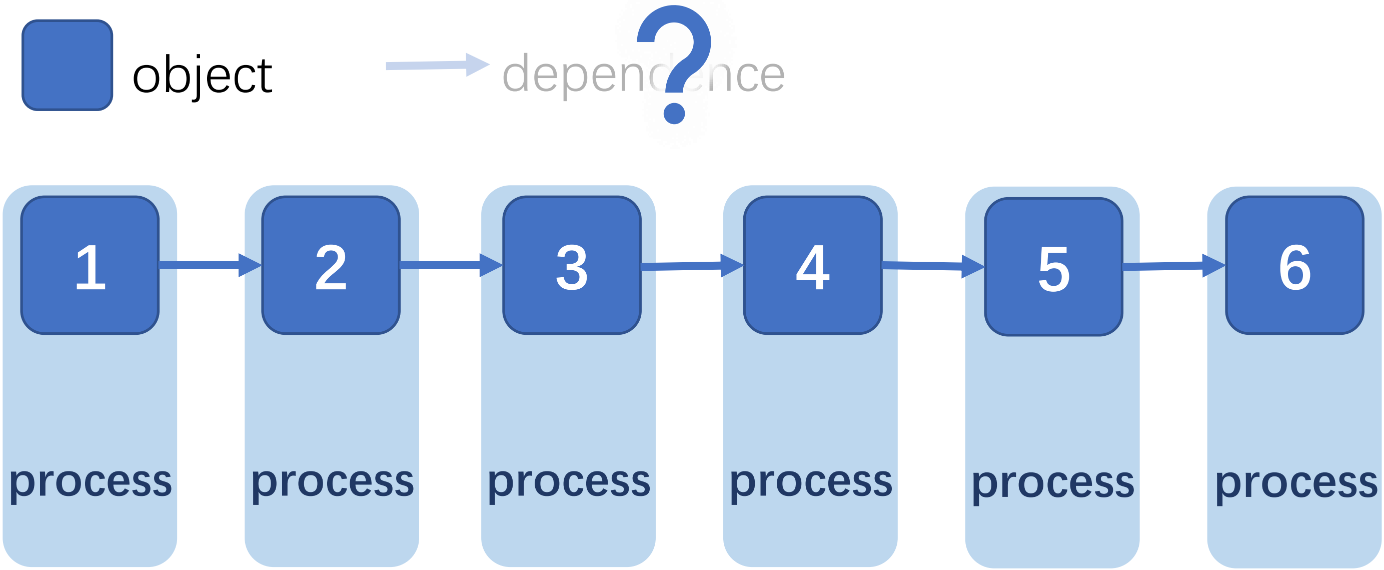
Sequential Iterative Algorithms

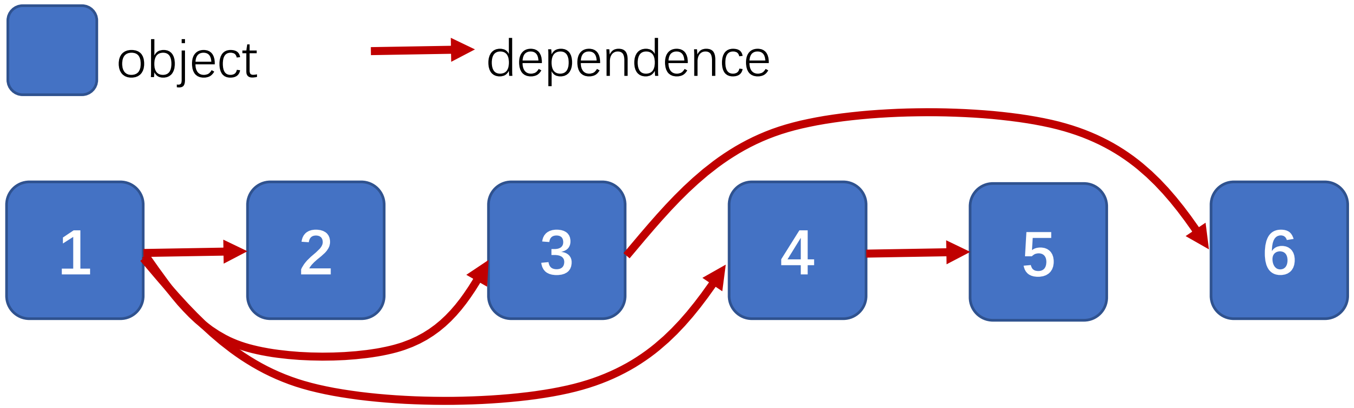
Process objects one by one in order




```
for i = 1 to n
    process object i
```

Parallelizing Sequential Iterative Algorithms

Identify the **dependences** among objects

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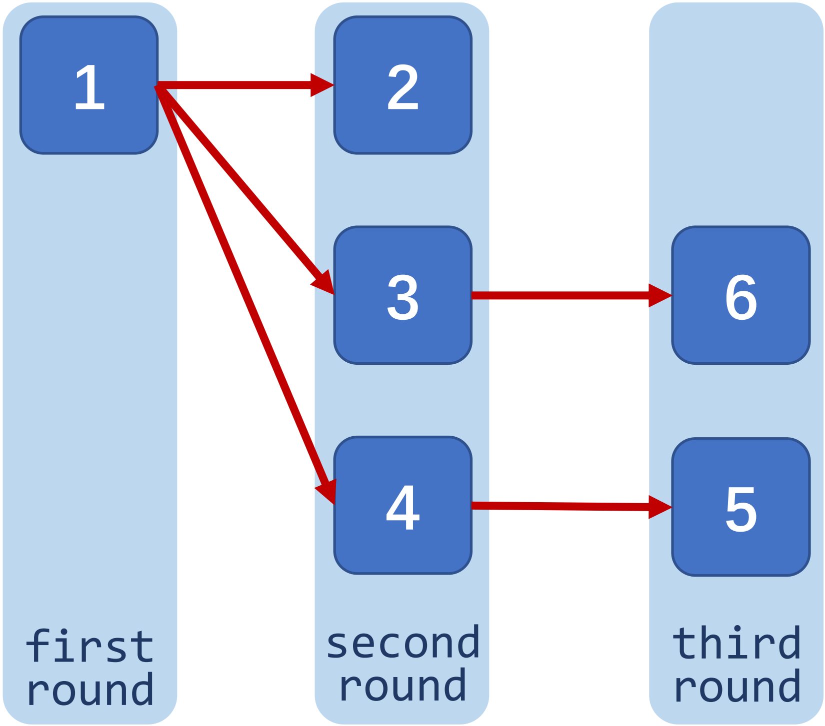
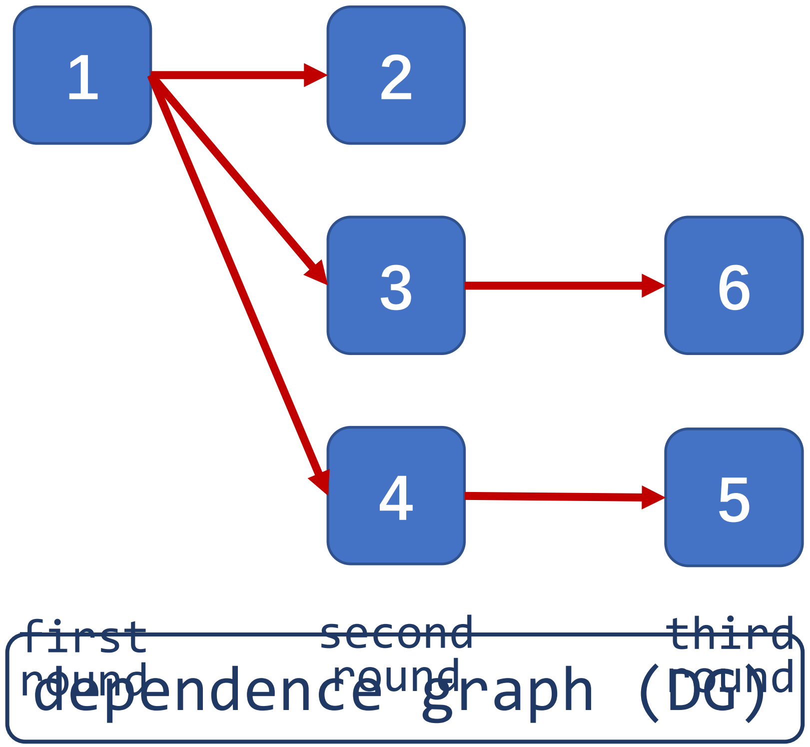
Parallelizing Sequential Iterative Algorithms

Obtain **parallelism**



Many Sequential Iterative Algorithms Can Be Parallel



G. Blelloch et al. (PPoPP 2012)

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W. Hasenplaugh et al. (SPAA 2014)

X. Pan et al. (NIPS 2015)

J. Shun et al. (SODA 2015)

M. Fischer and A. Noever. (SODA 2018)

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...

Ideal Parallel Algorithm

- **Parallelize vertices as much as possible**
- **Process a vertex only when it is ready**

Ideal Parallel Algorithm

- Round-efficiency - Parallelize vertices as much as possible
- Work-efficiency - Process a vertex only when it is ready

Efficiently Parallelizing Algorithms

Analysis is based on the **binary-forking model** (with TAS)

- Work: total number of operations
- Span: length of the longest execution path

Keep the algorithms efficient

- **Round-efficiency**
- **Work-efficiency**

Efficiently Parallelizing Algorithms

Analysis is based on the **binary-forking model** (with TAS)

- Work: total number of operations
- Span: length of the longest execution path

Keep the algorithms efficient

- **Round-efficiency**: $\tilde{O}(D)$ span (D=longest path length of the given DG)
- **Work-efficiency**:

Efficiently Parallelizing Algorithms

Analysis is based on the **binary-forking model** (with TAS)

- Work: total number of operations
- Span: length of the longest execution path

Keep the algorithms efficient

- **Round-efficiency**: not equivalent to optimal span
- **Work-efficiency**:
- Near work-efficiency: important for practical performance

Some algorithms could not be parallelized efficiently

Example: Longest Increasing Subsequence (LIS)

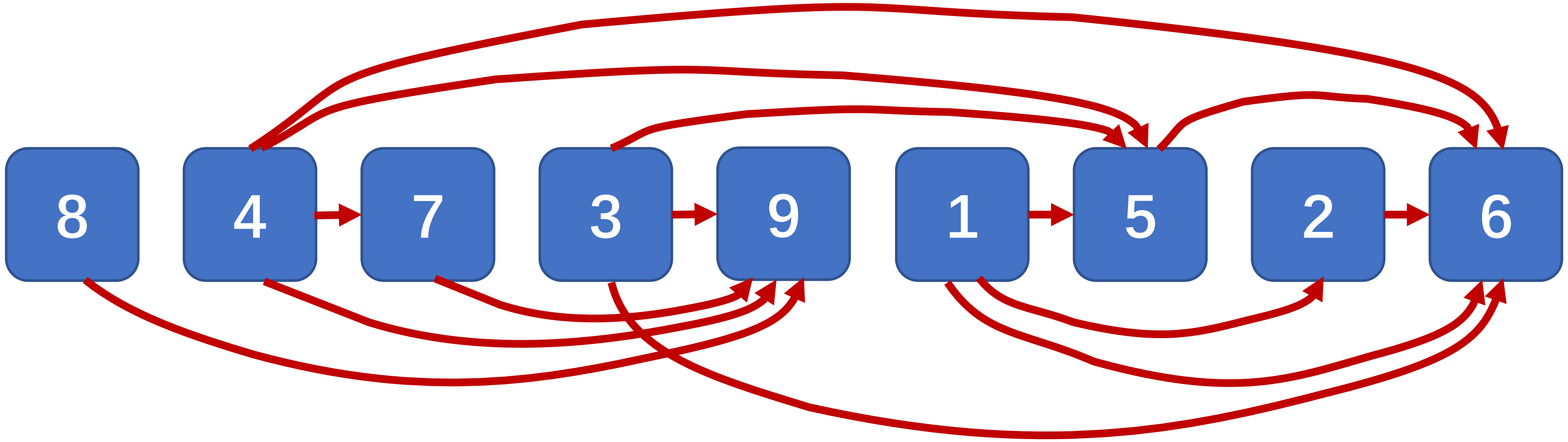
Given a sequence $s_1 \ldots s_n$

The LIS problem finds the longest subsequence s^* of s

Elements in s^* are strictly increasing

8	4	7	3	9	1	5	2	6

Example: Longest Increasing Subsequence (LIS)

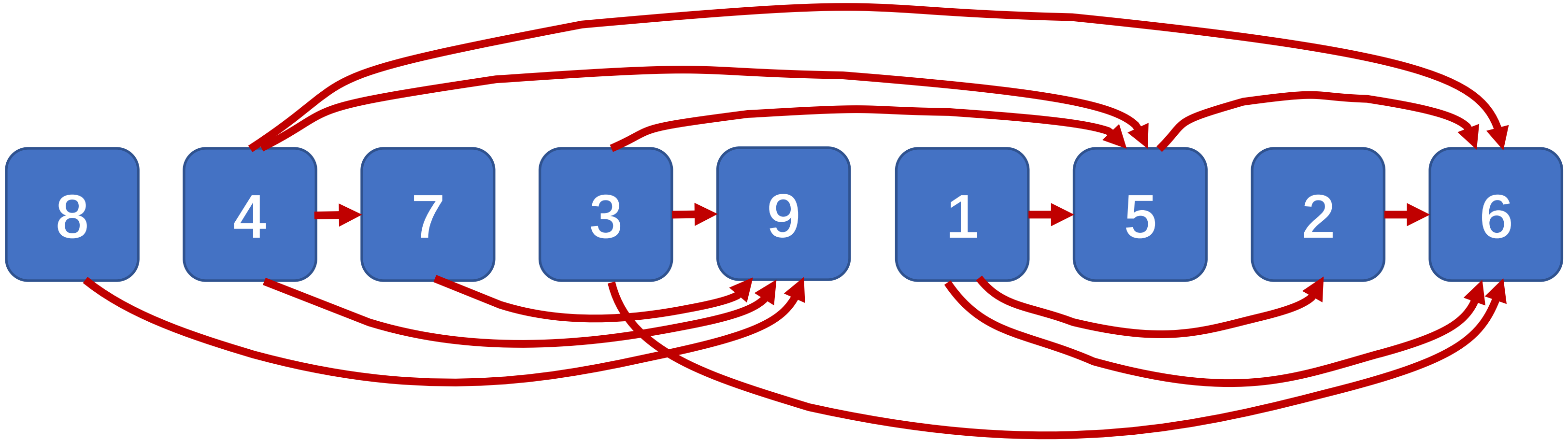


length of LIS ending with the i-th object

$$dp[i] = \max_{\boxed{j<i,\, s_j<s_i}} dp[j] + 1$$

Sequentially we can compute this in $\boldsymbol{O(n \log n)}$ cost

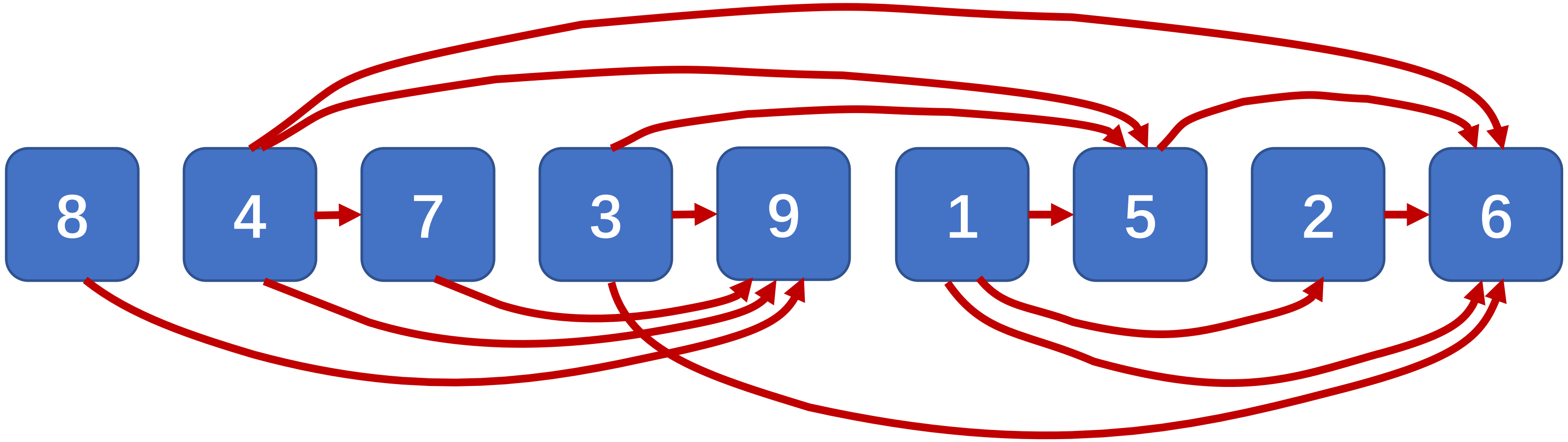
Example: Longest Increasing Subsequence (LIS)



Existing parallel algorithms are not nearly work-efficient or round efficient

- Galil et al.(Parallel Distrib 1994), Krusche, et al.(PPAM 2009), Nakashima et al. (ISPDC 2002), Semé, Thierry, et al. (ICCSA 2006), Thierry, et. al. (SPAA 2001) have $\Omega(n^{1.5})$ work
- Alam and Rahman's algorithm (IPL 2013) has $\Theta(n)$ span
- Krusche and Tiskin's algorithm (SPAA 2010) has $\tilde{O}(n \log^2 n)$ work and $\tilde{O}(n^{2/3})$ span

Example: Longest Increasing Subsequence (LIS)

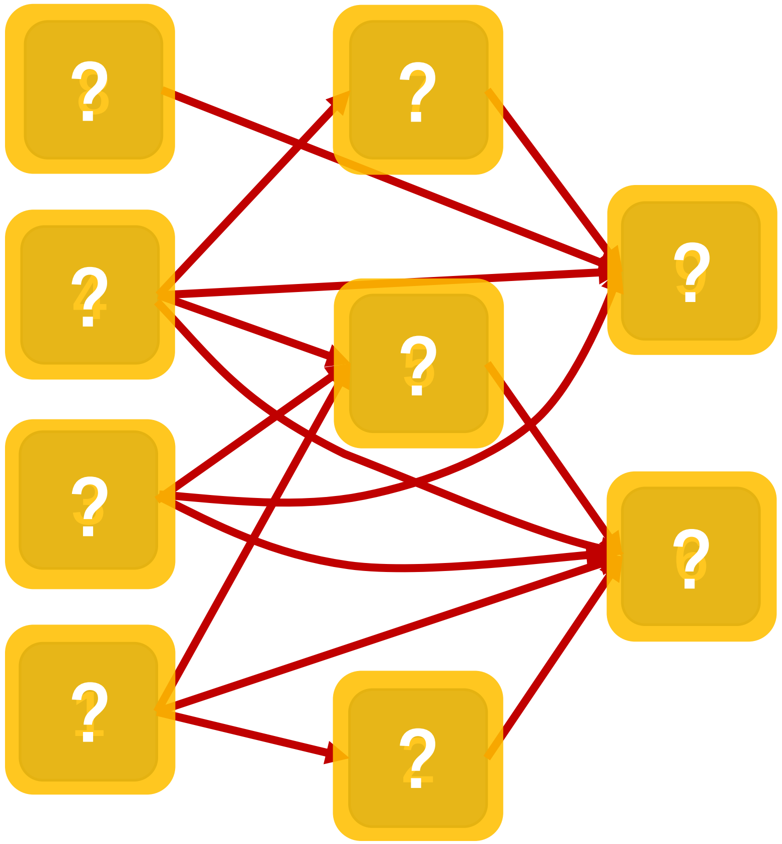


General approaches / frameworks

- Do not directly give efficient solutions to LIS

Deterministic Reservations

Proposed by Blelloch et al. (PPoPP 2012), used in algorithms from (Shun et al., SODA 2015)



In a high level:

- Access all unprocessed objects each round

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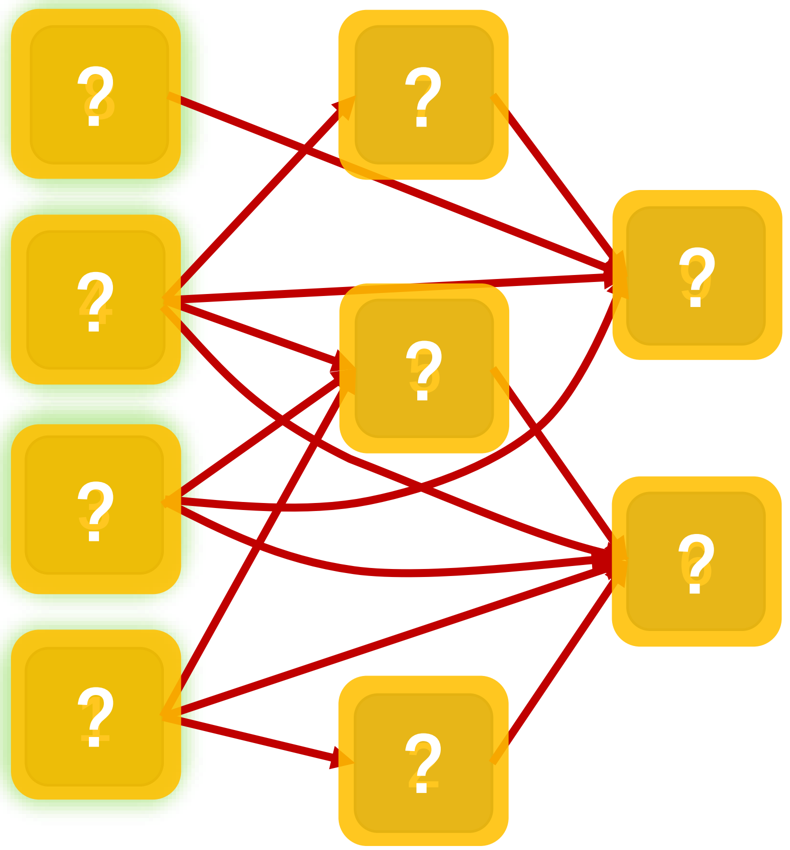

In a high level:

- Access all unprocessed objects each round
- Check their readiness

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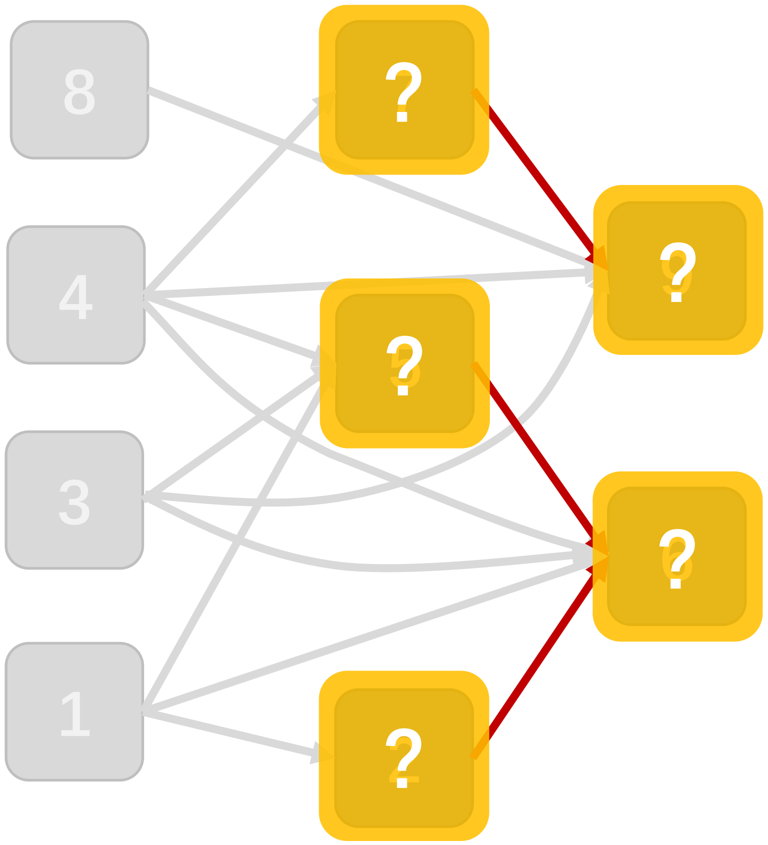

In a high level:

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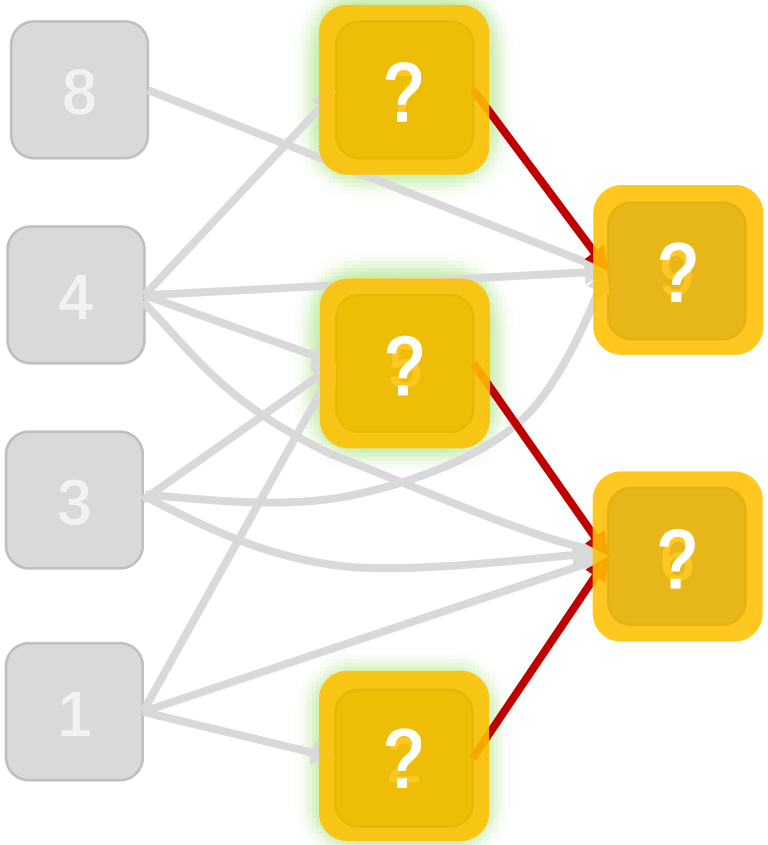

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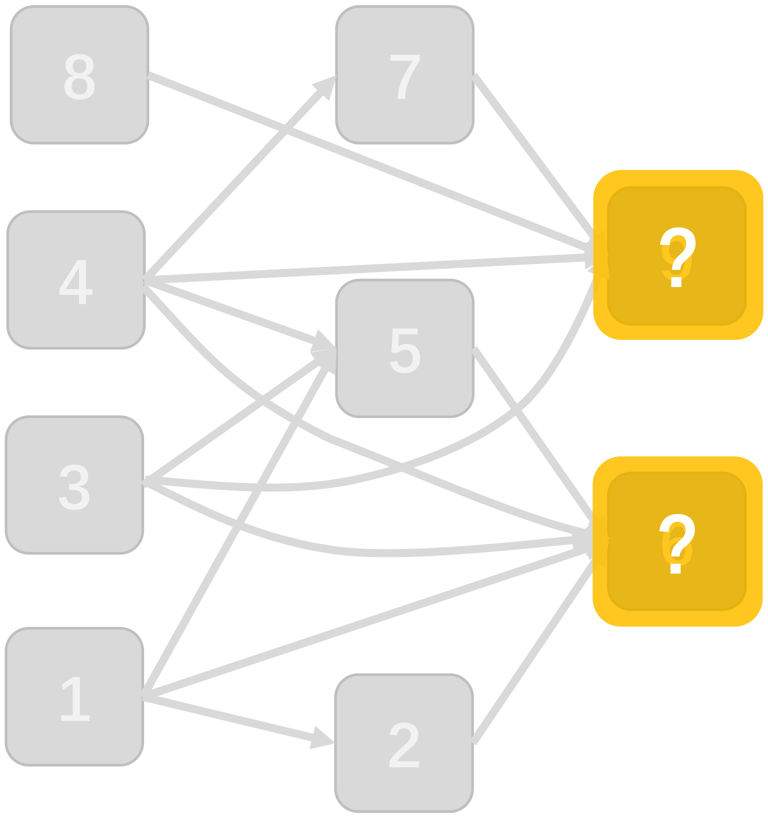

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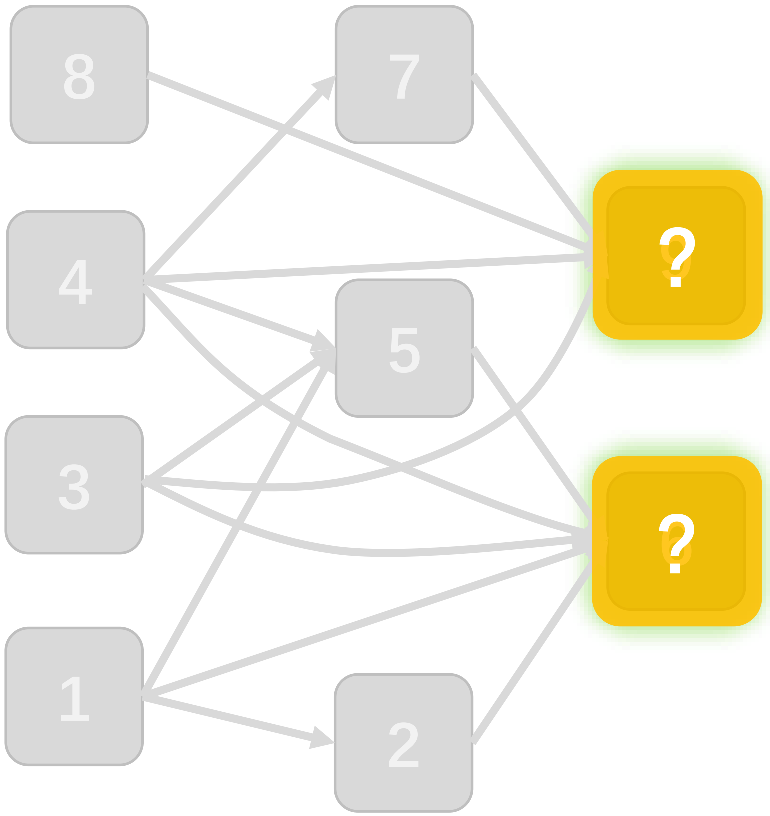

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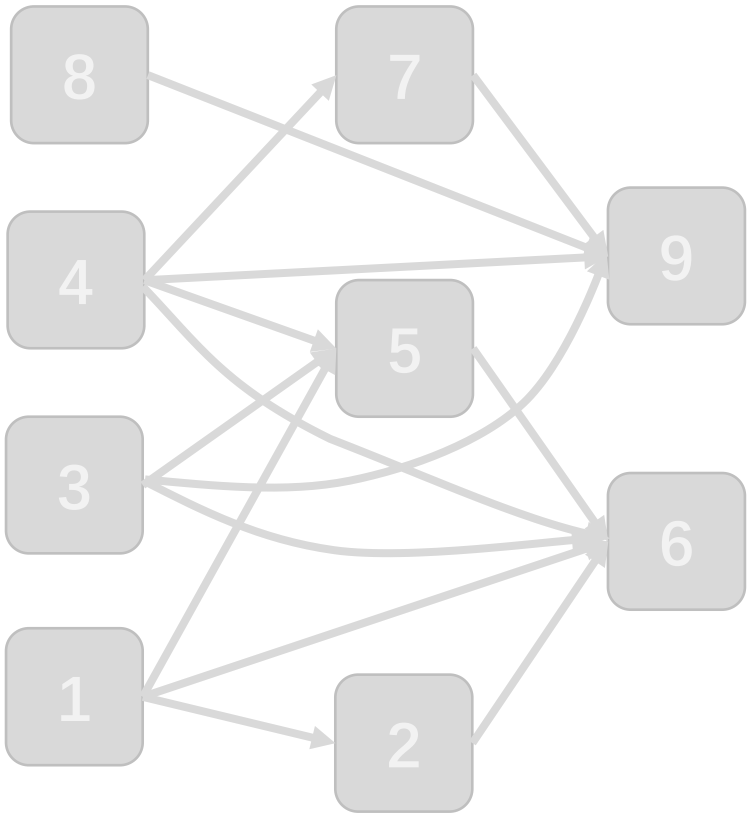


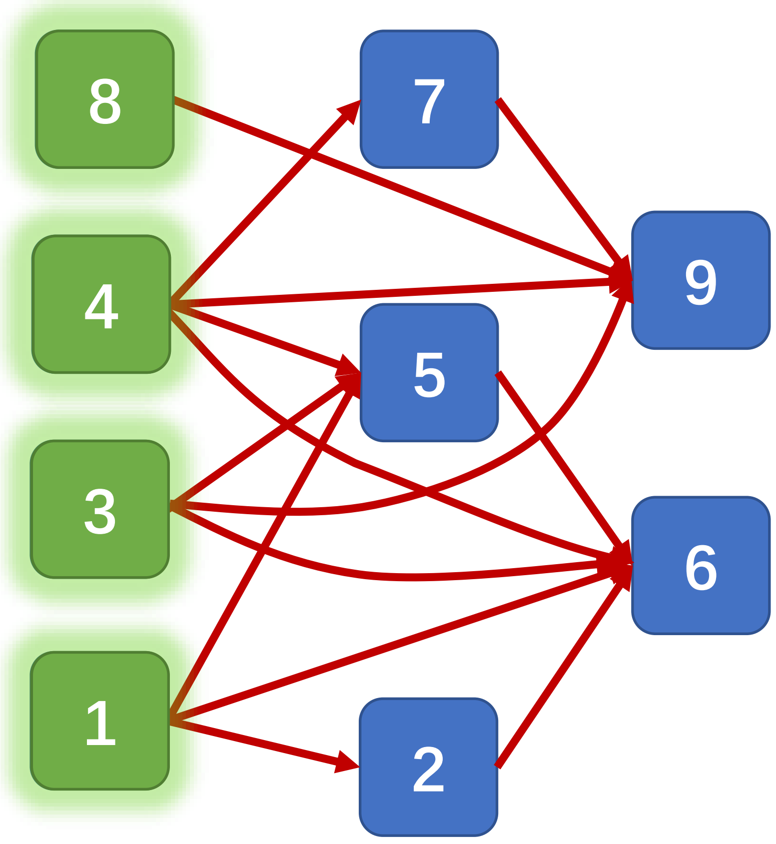
In a high level:

- Access all unprocessed objects each round
- Check their readiness
- Process the ready objects

- Work-efficient only when the work decreases geometrically in every round
- $\Theta(\mathbf{round} \cdot n)$ work for LIS
- $O(n^2)$ worst-case

Activation-based Approaches

Used in Blelloch et al.(SPAA 2012, SPAA 2020), M. Fischer and A. Noever (SODA 2018) , generalized by Blelloch et al. (SPAA 2020)



- Activate some successors based on the edges in the DG

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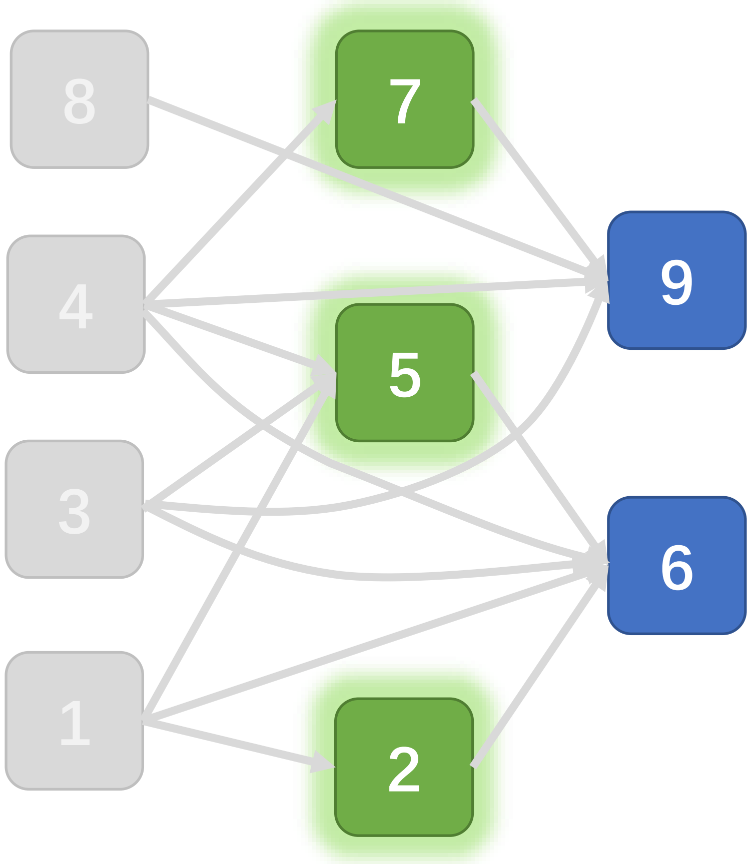

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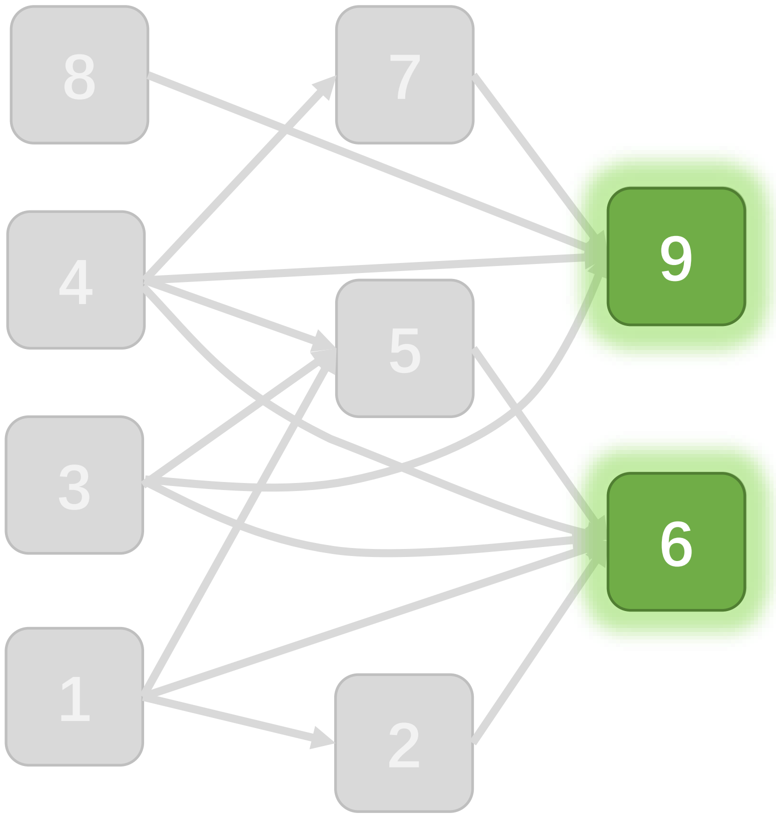

- Activate some successors based on the edges in the DG
- Process the ready ones

- Go through all the edges
- Take $\Theta(m)$ work for LIS (m =#edges in the DG)
- m can be up to $\Theta(n^2)$

	Cost of single readiness check	# candidates in next round
Deterministic reservations	can be fast	all the rest
Activation-based	$\Theta(\mathbf{deg})$	only successors

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This Work

Phase-parallel framework to analyze dependences

- Core concept: $\boxed{\textbf{rank}}$
- Vertex-centric manner: avoid checking all the edges

Two general techniques to design algorithms in this framework

- Type 1 (more interesting problem in our paper)
- Type 2 (LIS problem is here)

Phase-parallel Framework

A general technique to parallelize the iterative algorithms

Formally define **rank** from the independence system $(S, \mathcal{F})$

- Feasible set: $\mathcal{F}(x) = \{E \in \mathcal{F} : E \subseteq x^{\downarrow}, x \in E\}$
- Maximum feasible set: $\mathrm{MFS}(x) = \arg\max_{E \in \mathcal{F}(x)} |E|$
- Rank: $\mathrm{rank}(x) = |\mathrm{MFS}(x)|$

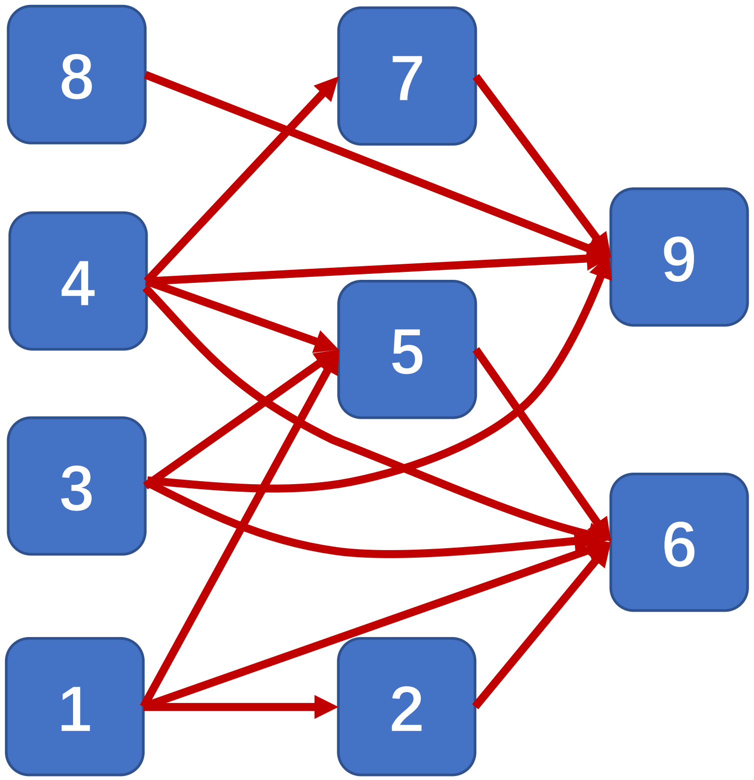
Apply to LIS

$$\mathcal{F}(x) = \{E \in \mathcal{F}: E \subseteq x^{\downarrow}, x \in E\}$$

$$\mathrm{MFS}(x) = \arg\max_{E \in \mathcal{F}(x)} |E|$$

$$\mathrm{rank}(x) = |\mathrm{MFS}(x)|$$

Apply to LIS



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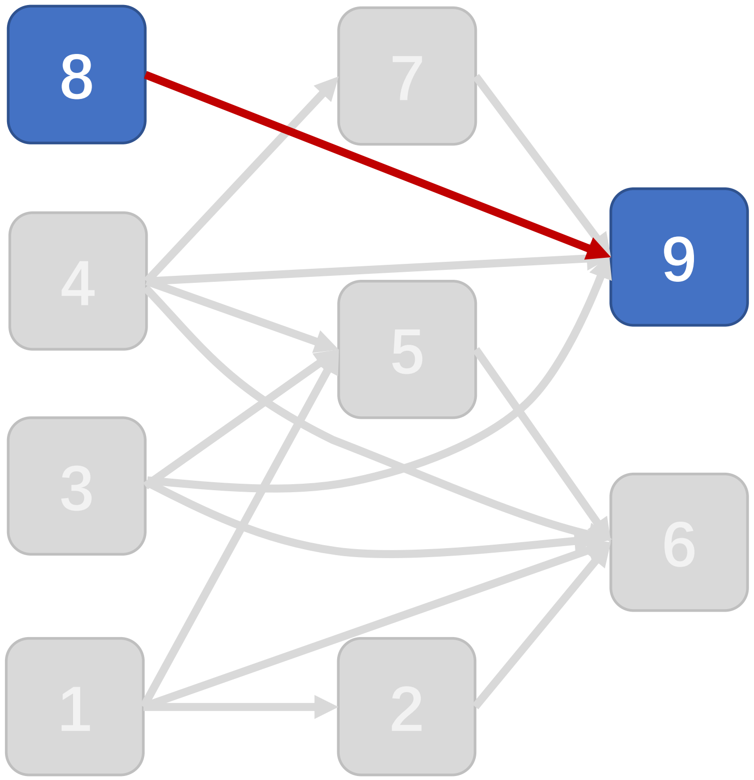

Given an object x

- $\mathcal{F}(x) = \{E \in \mathcal{F}: E \subseteq x^{\downarrow}, x \in E\}$
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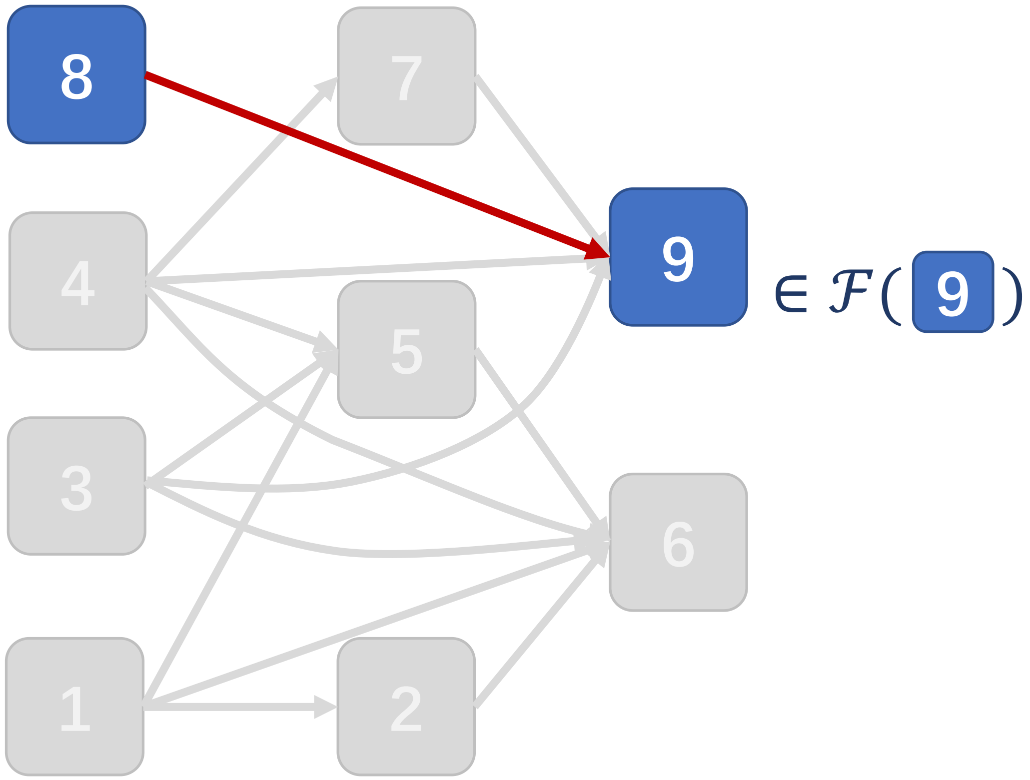
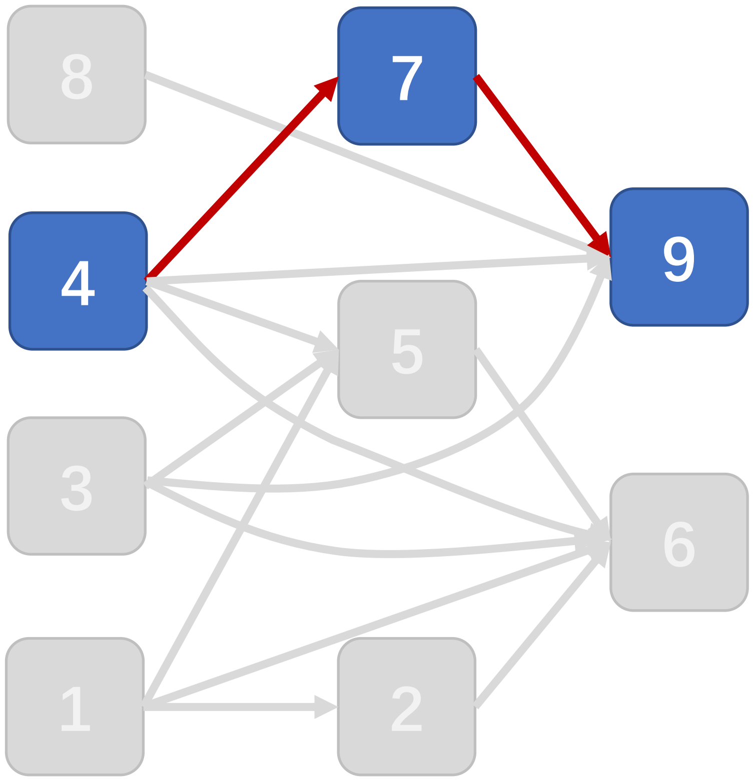
Apply to LIS



Given an object x

- $\mathcal{F}(x)$ increasing subsequence ending with object x
- $\mathrm{MFS}(x) = \arg\max_{E \in \mathcal{F}(x)} |E|$
- $\mathrm{rank}(x) = |\mathrm{MFS}(x)|$

Apply to LIS

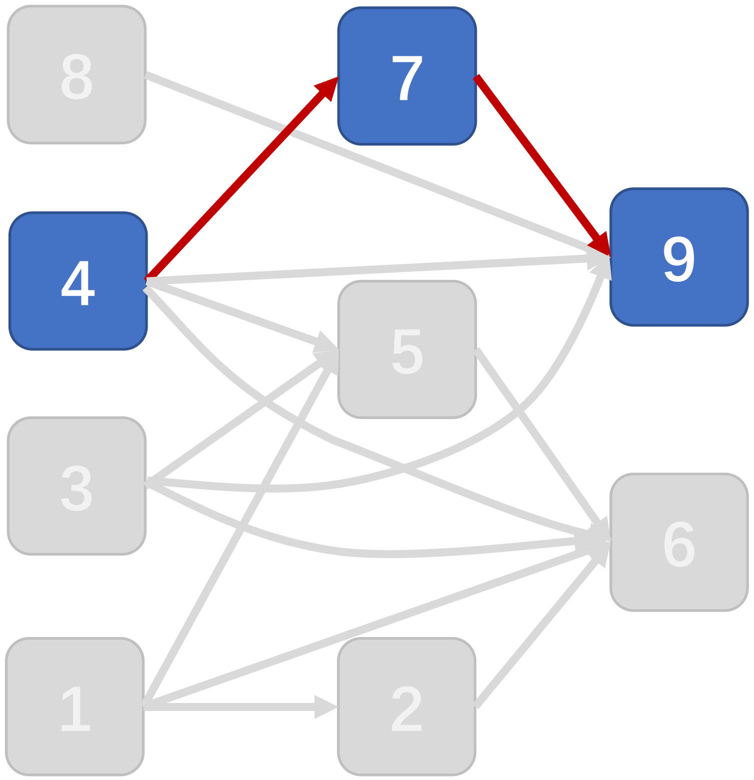


$= \mathrm{MFS}(\boxed{9})$

Given an object x

- $\mathcal{F}(x)$ — increasing subsequence ending with object x
- $\mathrm{MFS}(x)$ — LIS ending with object x
- $\mathrm{rank}(x) = |\mathrm{MFS}(x)|$

Apply to LIS

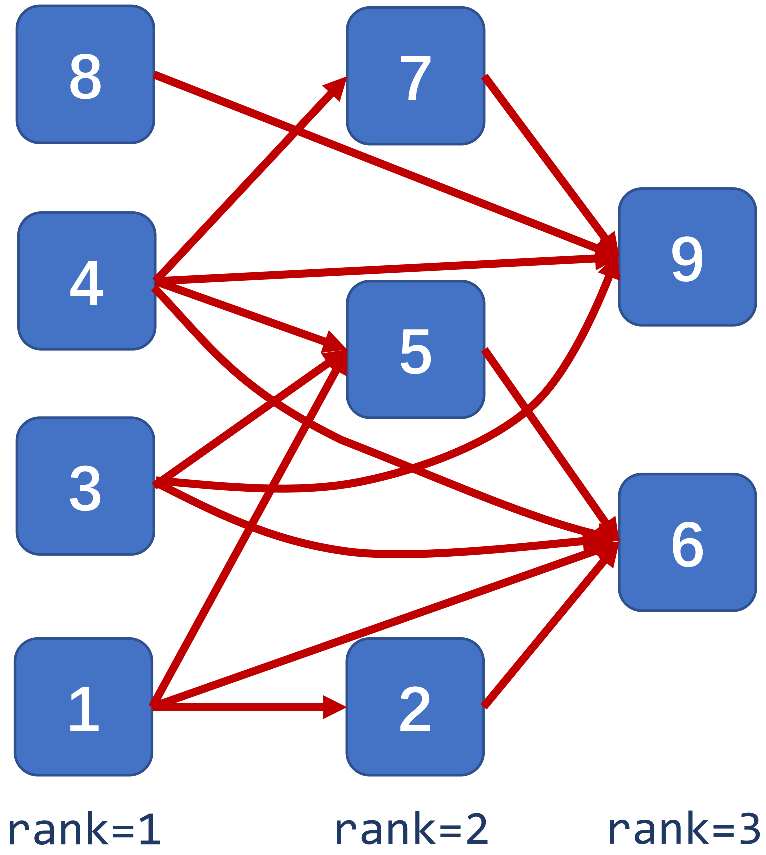


rank(9)=3

Given an object x

- $\mathcal{F}(x)$ — increasing subsequence ending with object x
- $\mathrm{MFS}(x)$ — LIS ending with object x
- $\mathrm{rank}(x)$ — the length of LIS

Apply to LIS



Given an object x

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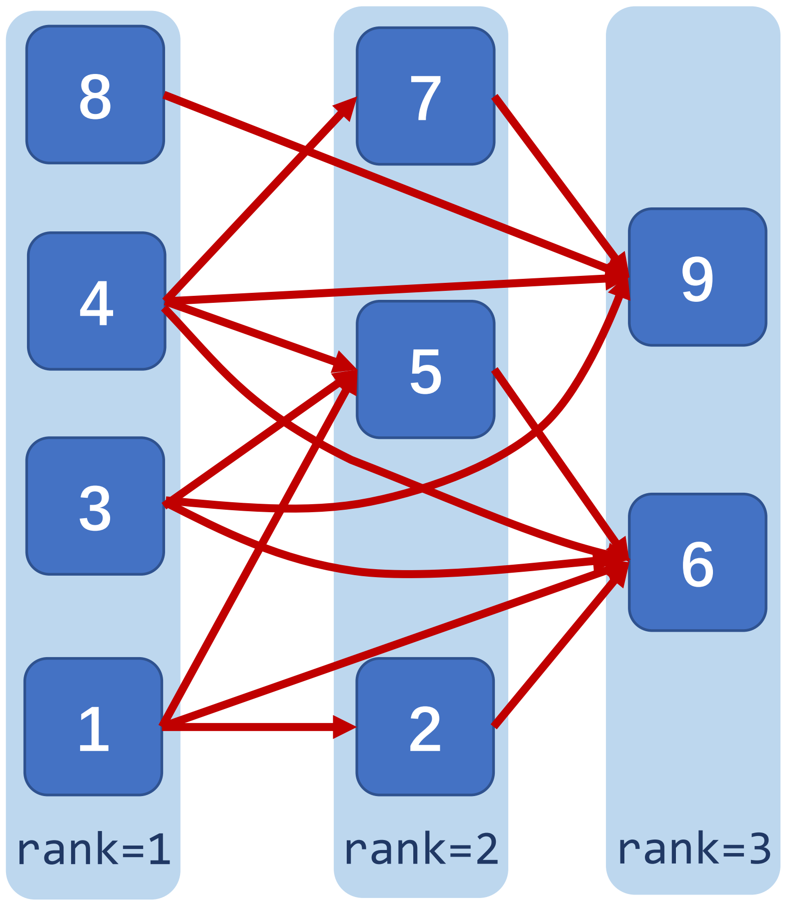
The **earliest "phase"** when an object is ready

With certain conditions, the **rank** of an object is its depth in DG

Processes objects in the order of ranks



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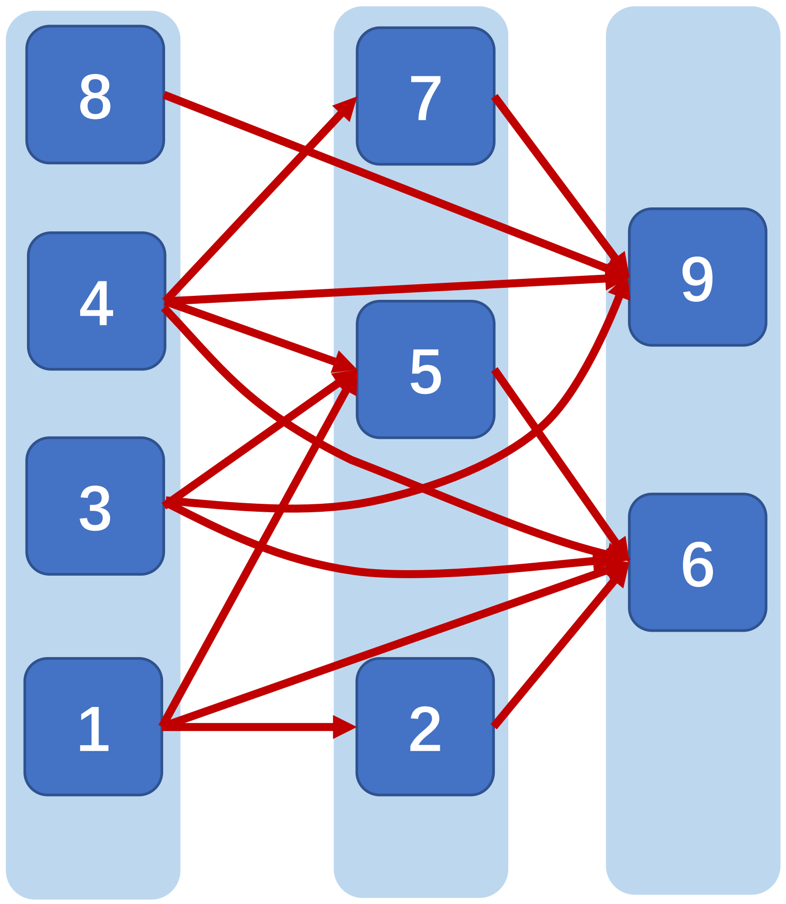


Challenge

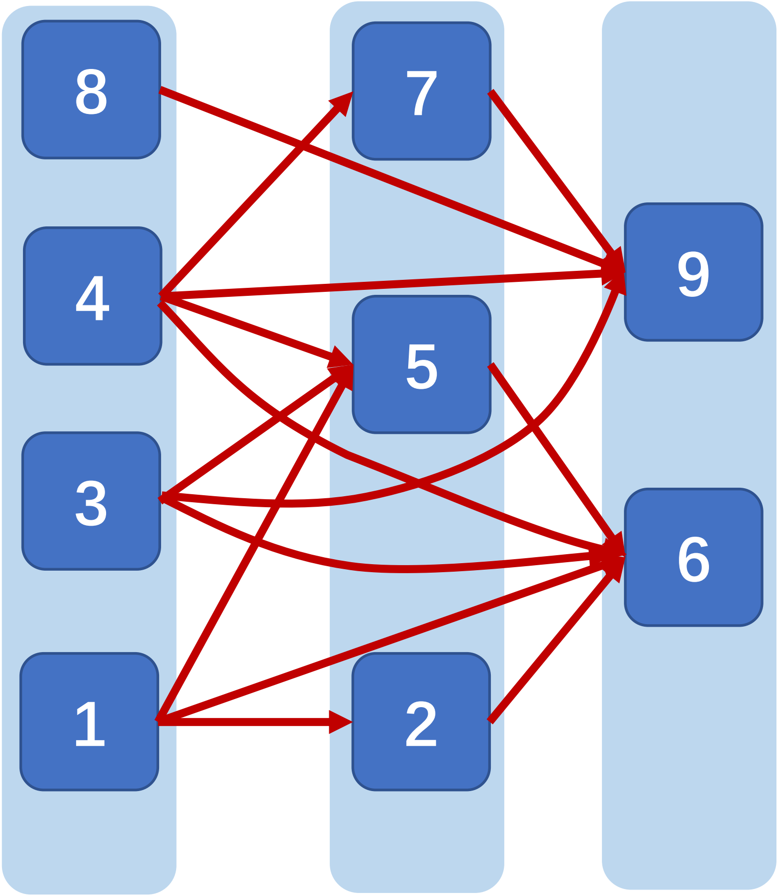
Find the ready objects in each round

Avoid visiting all edges

Solution

“Vertex-centric” approach

Core idea: select a "pivot" for every object



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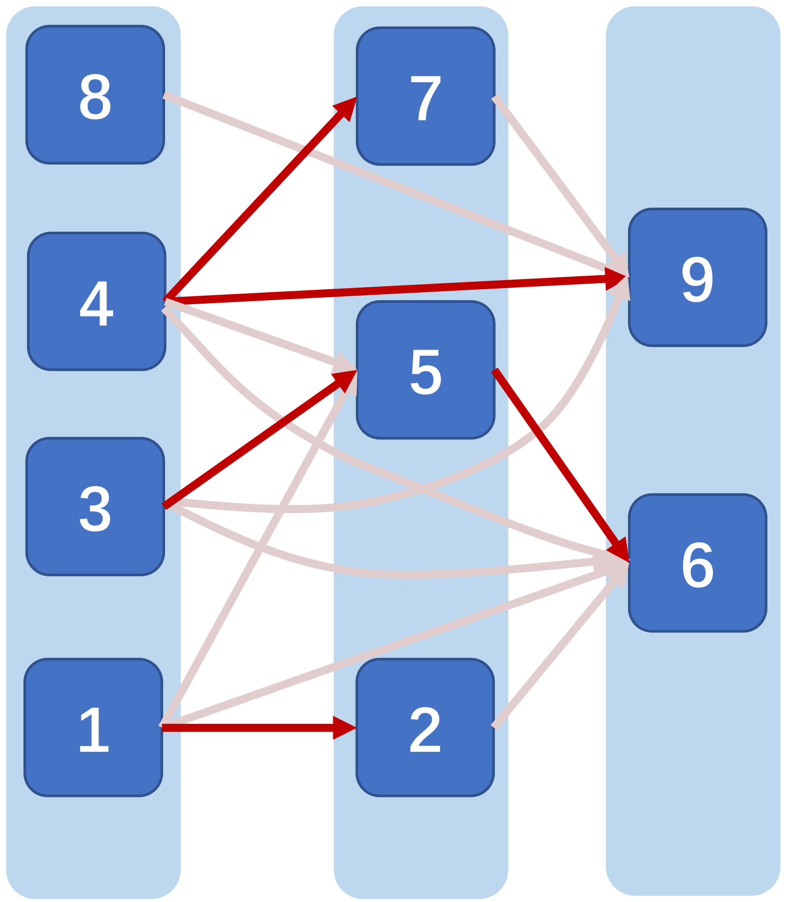
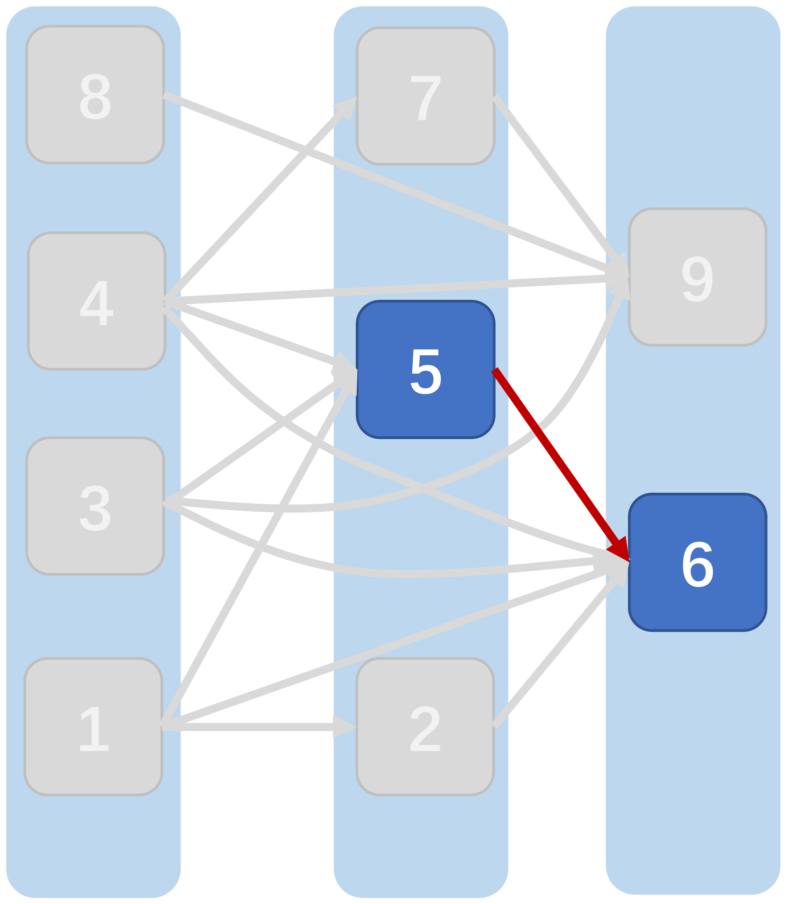


Pivot: an unfinished predecessor selected at uniformly random

If the pivot of an object hasn't finished, this object cannot be ready

We check readiness of an object **only when its pivot finishes**

Core idea: select a “pivot” for every object



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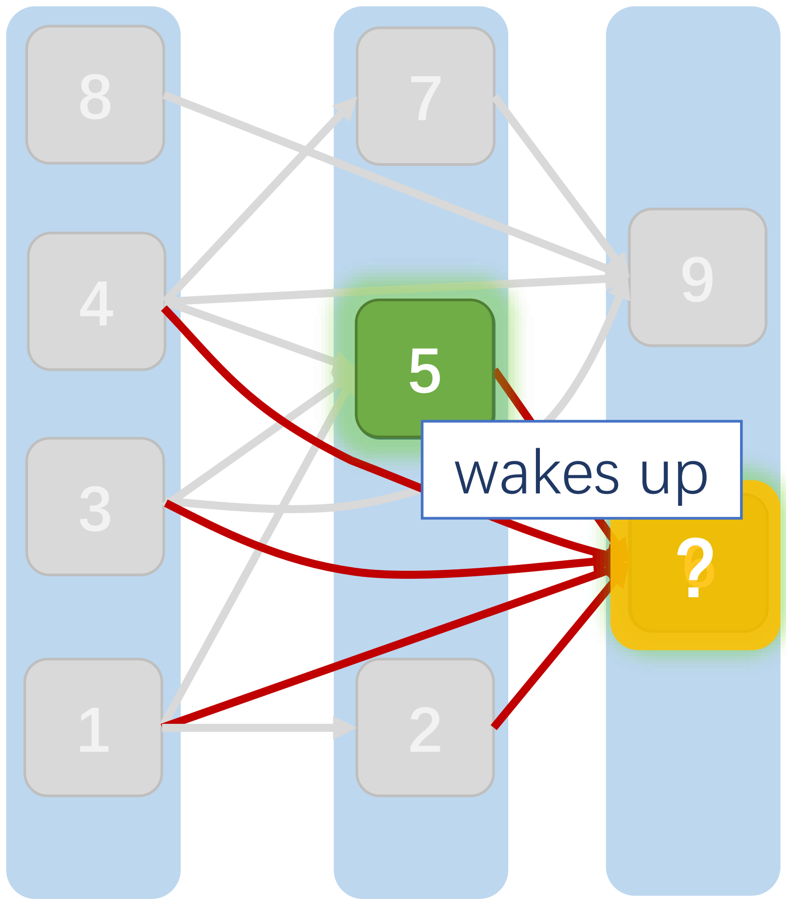

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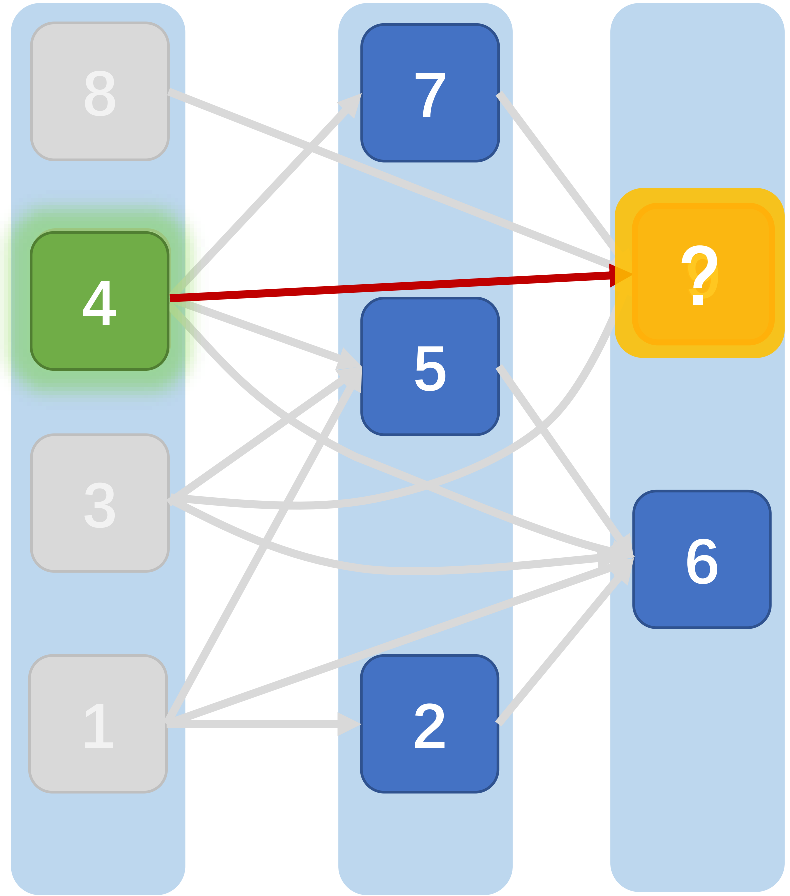
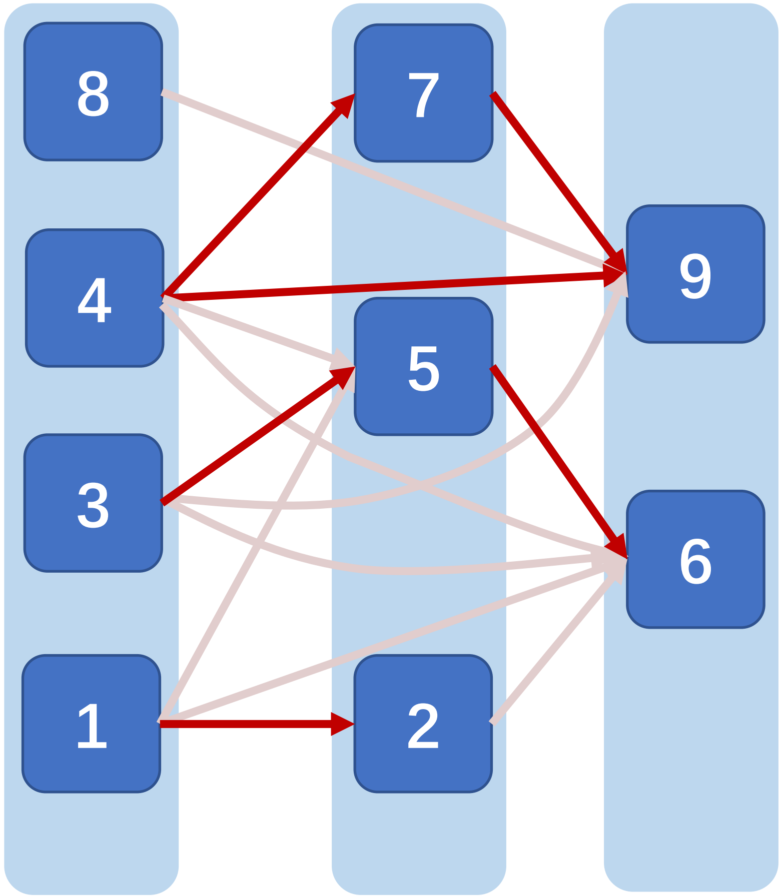
Wake-up and re-pivoting



If the object waked up is **not ready**

- Update pivot to another unprocessed objects
- Sleep until being waked up the next time

Wake-up and re-pivoting



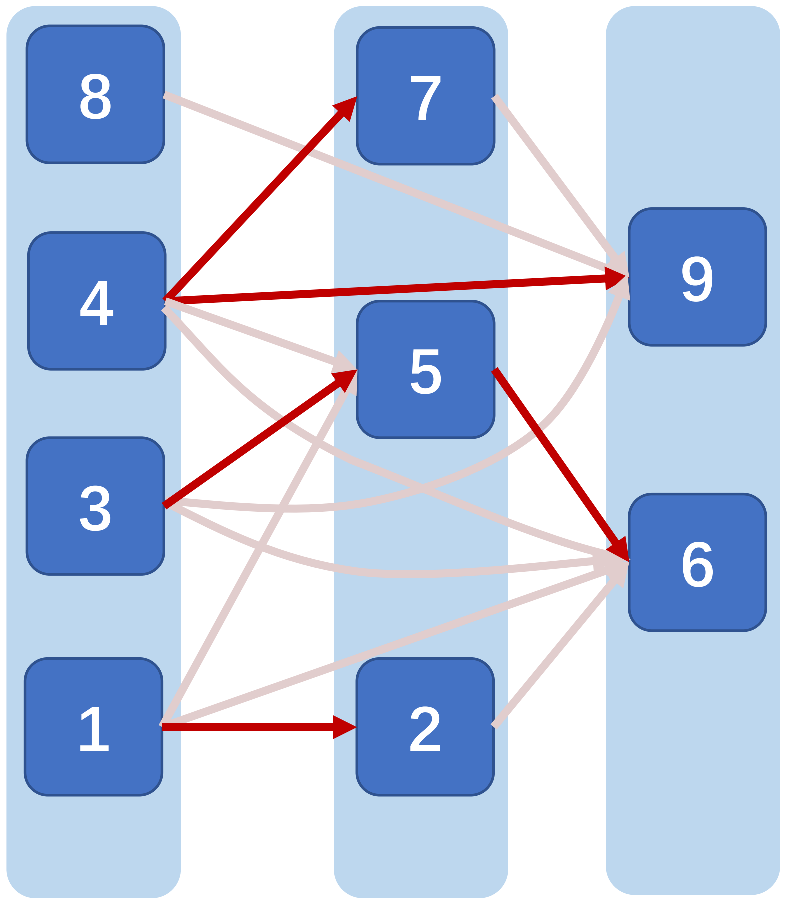
Assign a **pivot** to each object

Choose another pivot if not ready when being waked up

Save work as only a few edges are evaluated

If choosing pivots at uniformly random,
#evaluated edges is $O(n \log n)$ w.h.p.

Apply to the LIS algorithm



$\mathtt{F = \{ready\ at\ the\ beginning\}}$

while $F \neq \emptyset$

{

process F

$S_{woken} = \{\text{objects woken by F}\}$

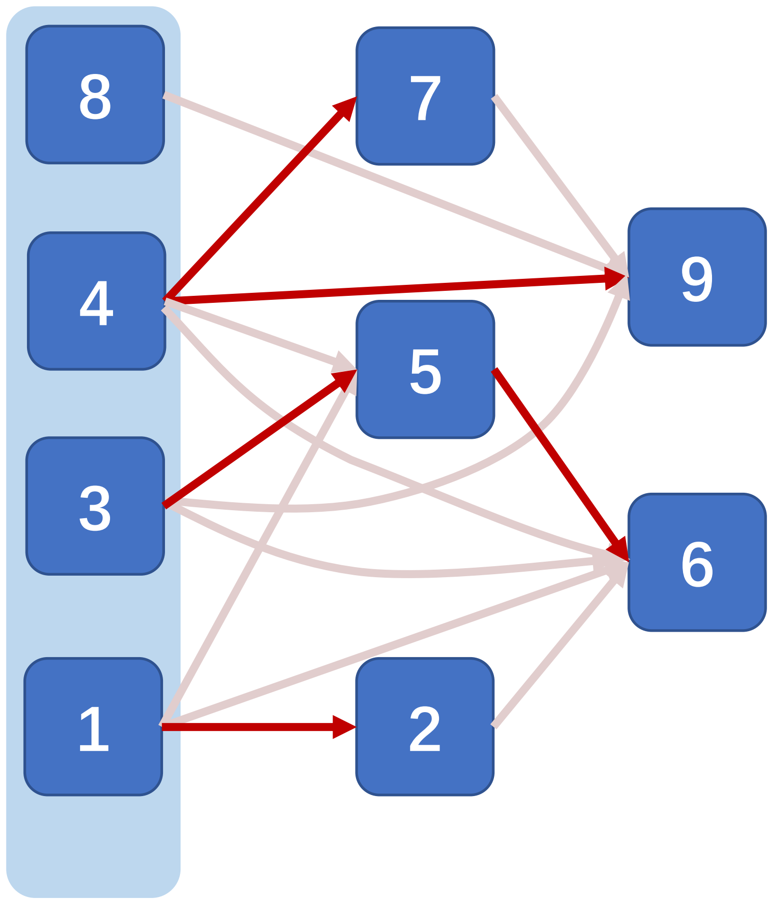
$S_{ready} = \{u \in S_{woken},\ u \text{ is ready}\}$

update pivots in $S_{woken} - S_{ready}$

$F = S_{ready}$

}

Apply to the LIS algorithm



F = {ready at the beginning}

while F $\neq \emptyset$

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    S_{woken} = {objects woken by F}

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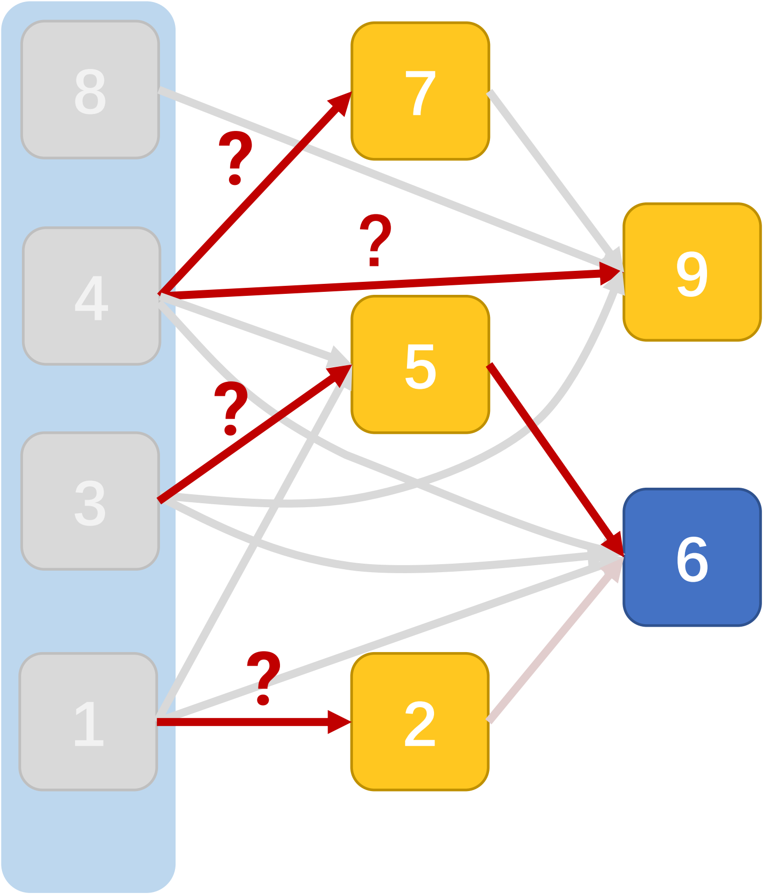
    F = S_{ready}

}

Apply to the LIS algorithm




```
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```
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```
$S_{woken} - S_{ready}$

$F = S_{ready}$

```
}
```

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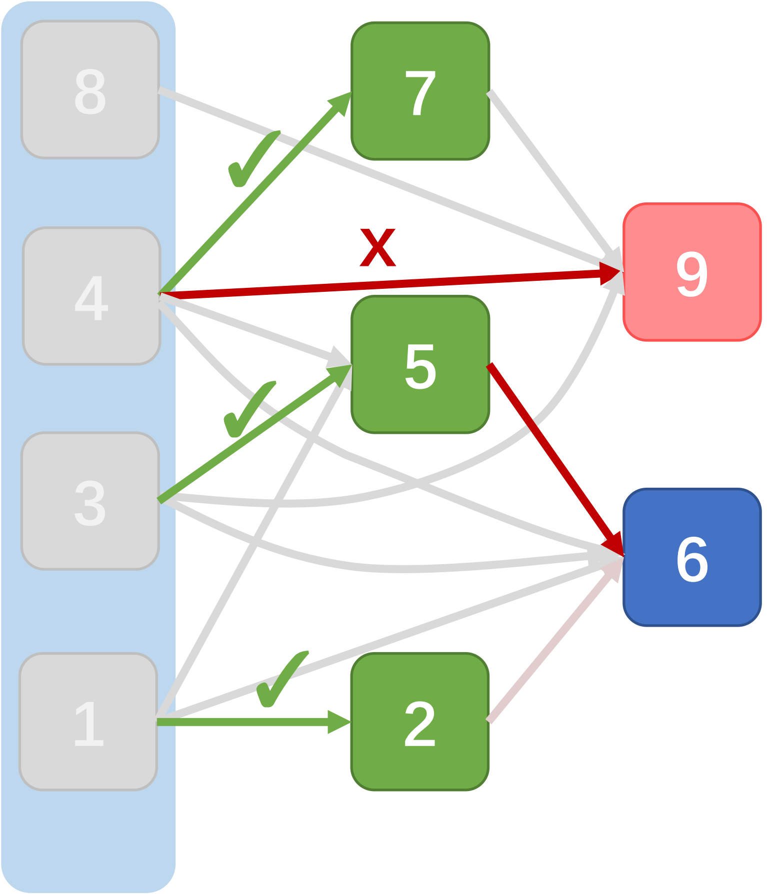

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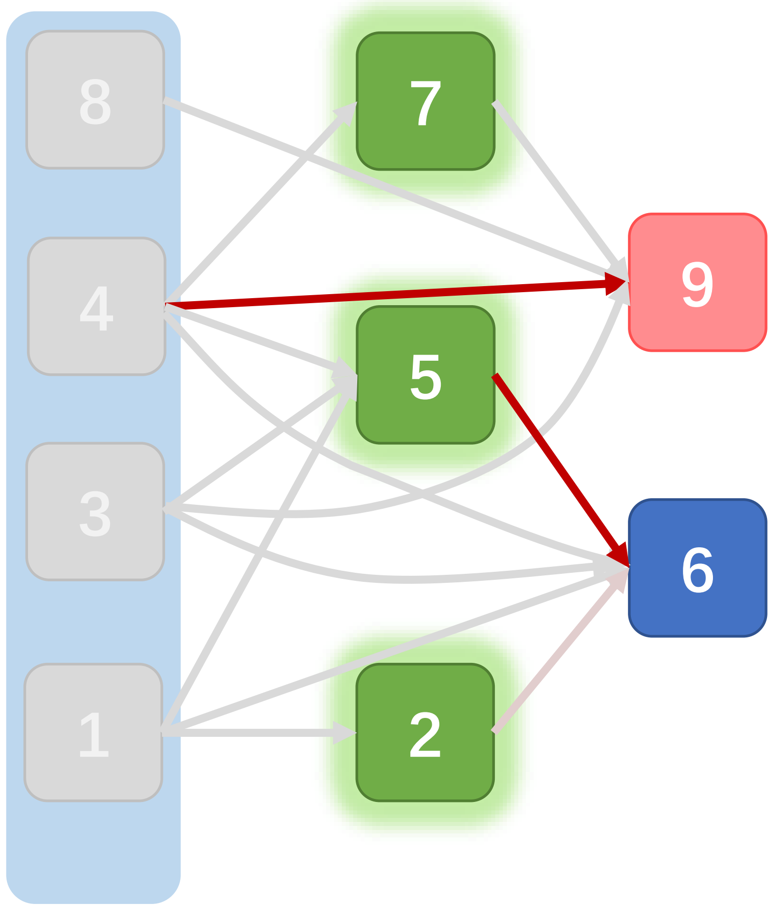

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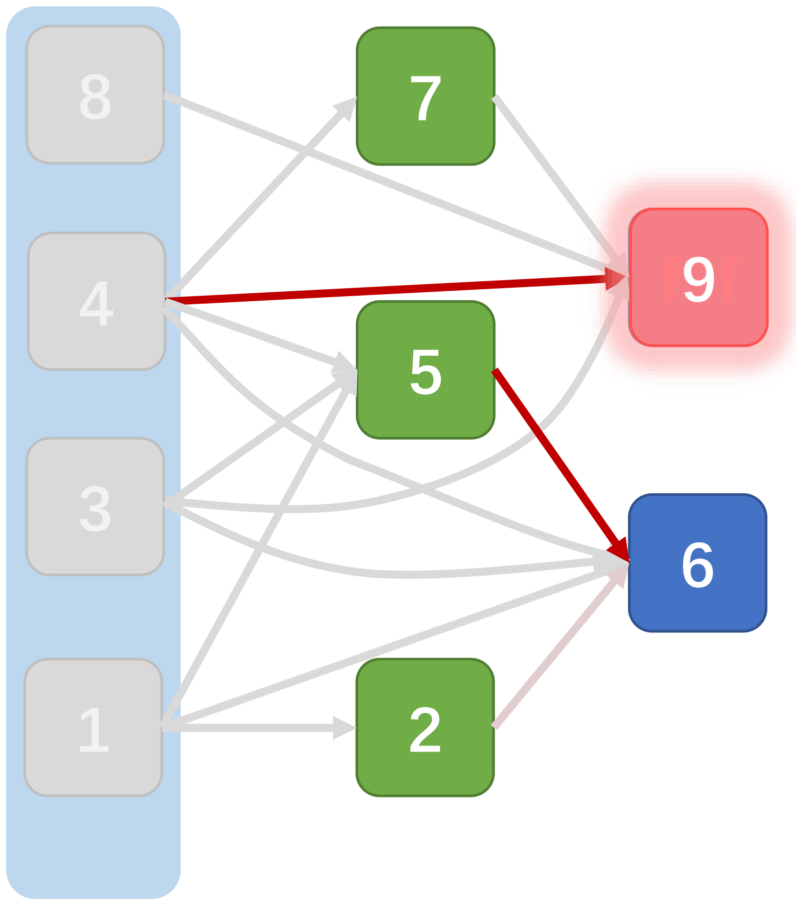

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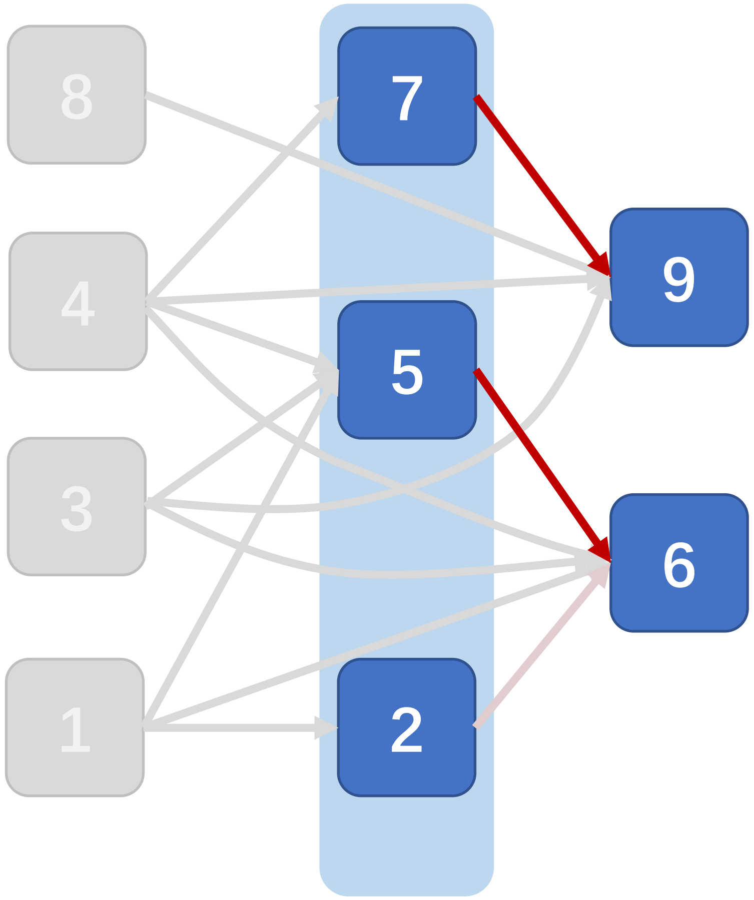

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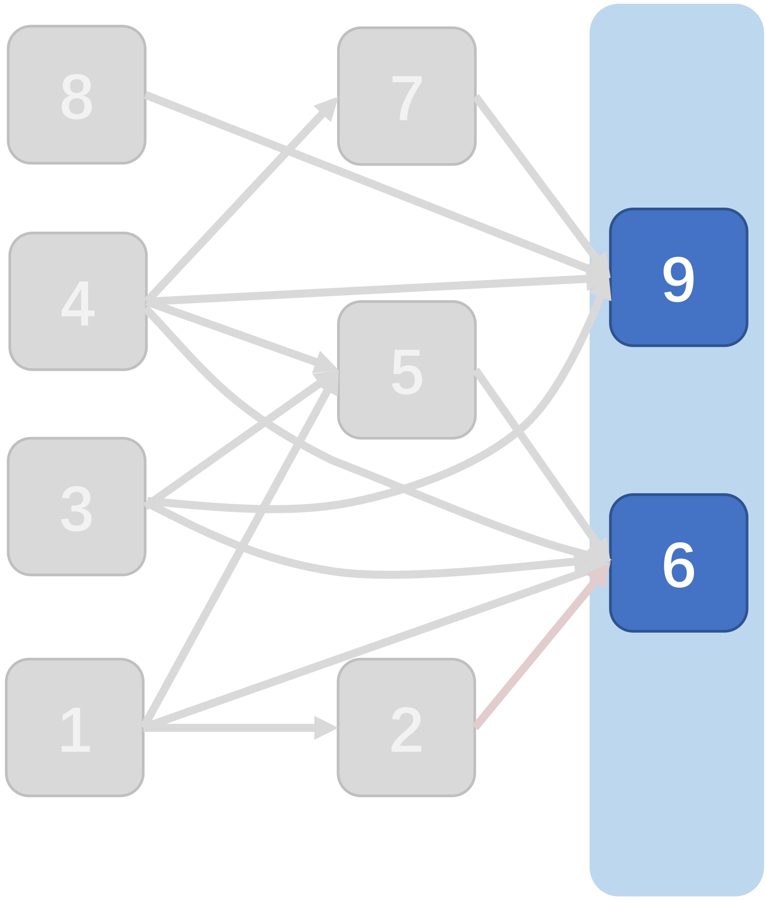

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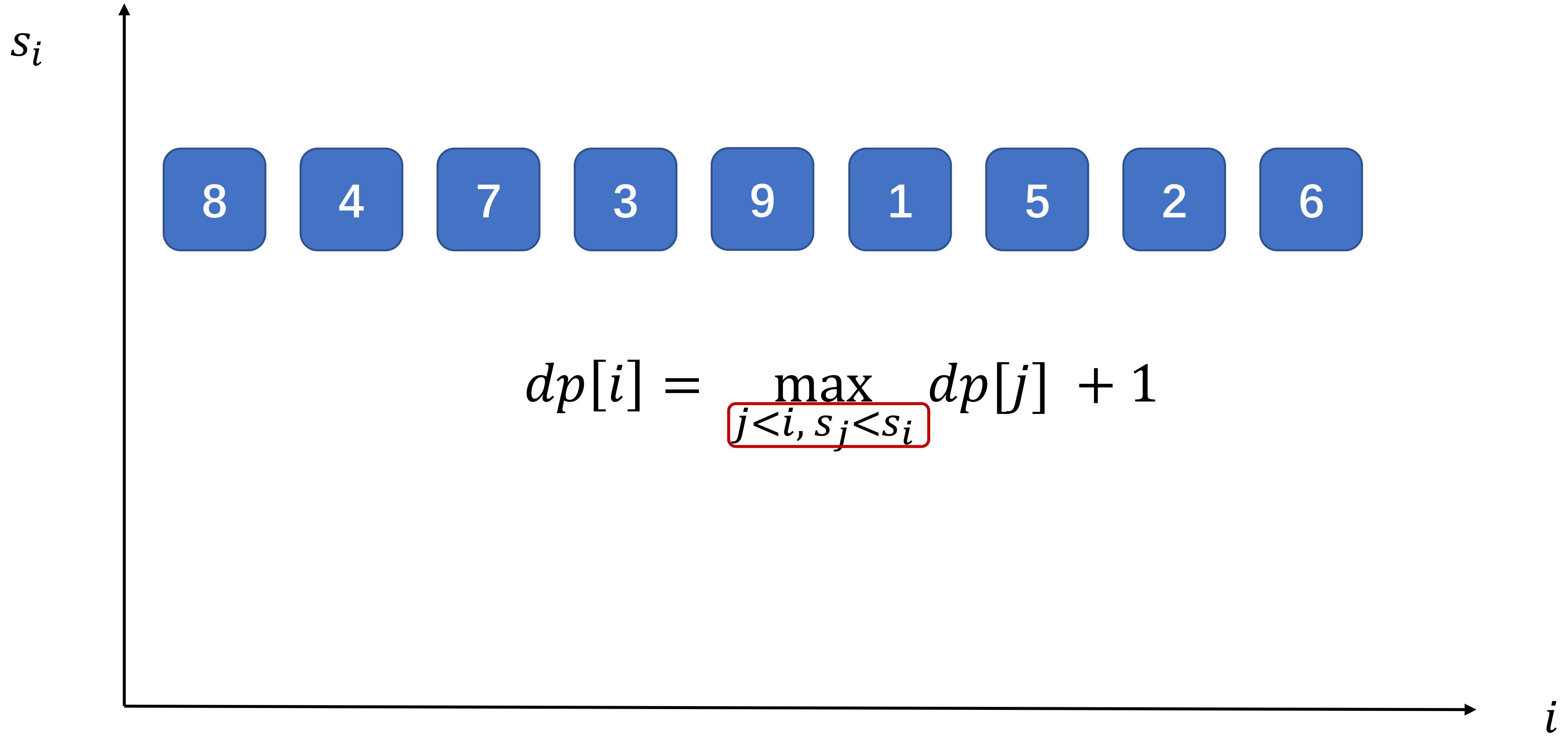
    $F = S_{ready}$
}

Apply to the LIS algorithm

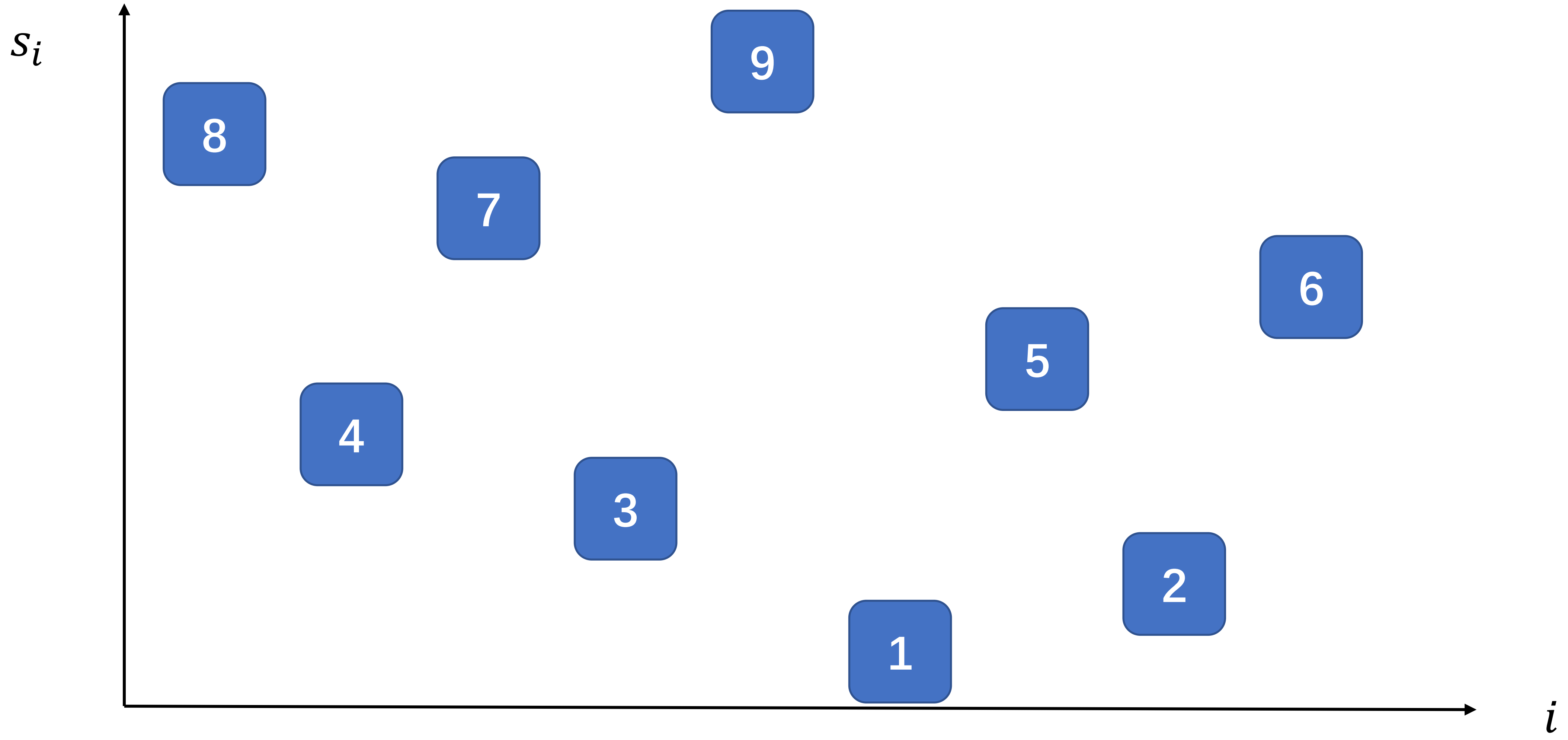
	F = {ready at the beginning}
	while $F \neq \emptyset$
	{
DP value computation	process F
	$S_{woken} = \{\text{objects woken by F}\}$
Readiness check	$S_{ready} = \{u \in S_{woken},\ u \text{ is ready}\}$
Pivot reselection	update pivots in $S_{woken} - S_{ready}$
	$F = S_{ready}$
	}

Vertex-centric Approach

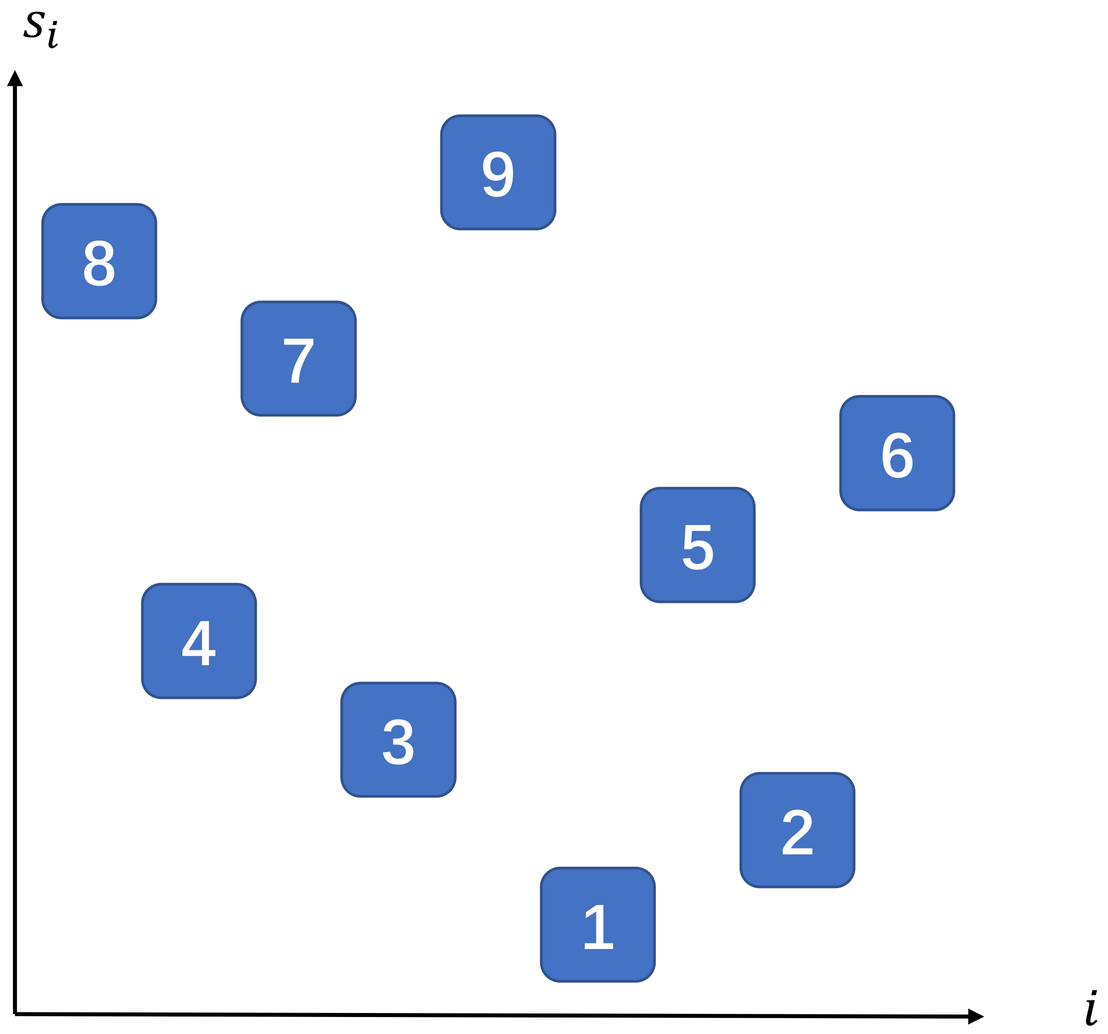


$$dp[i] = \max_{\boxed{j<i,\, s_j<s_i}} dp[j] + 1$$

Vertex-centric Approach



Vertex-centric Approach



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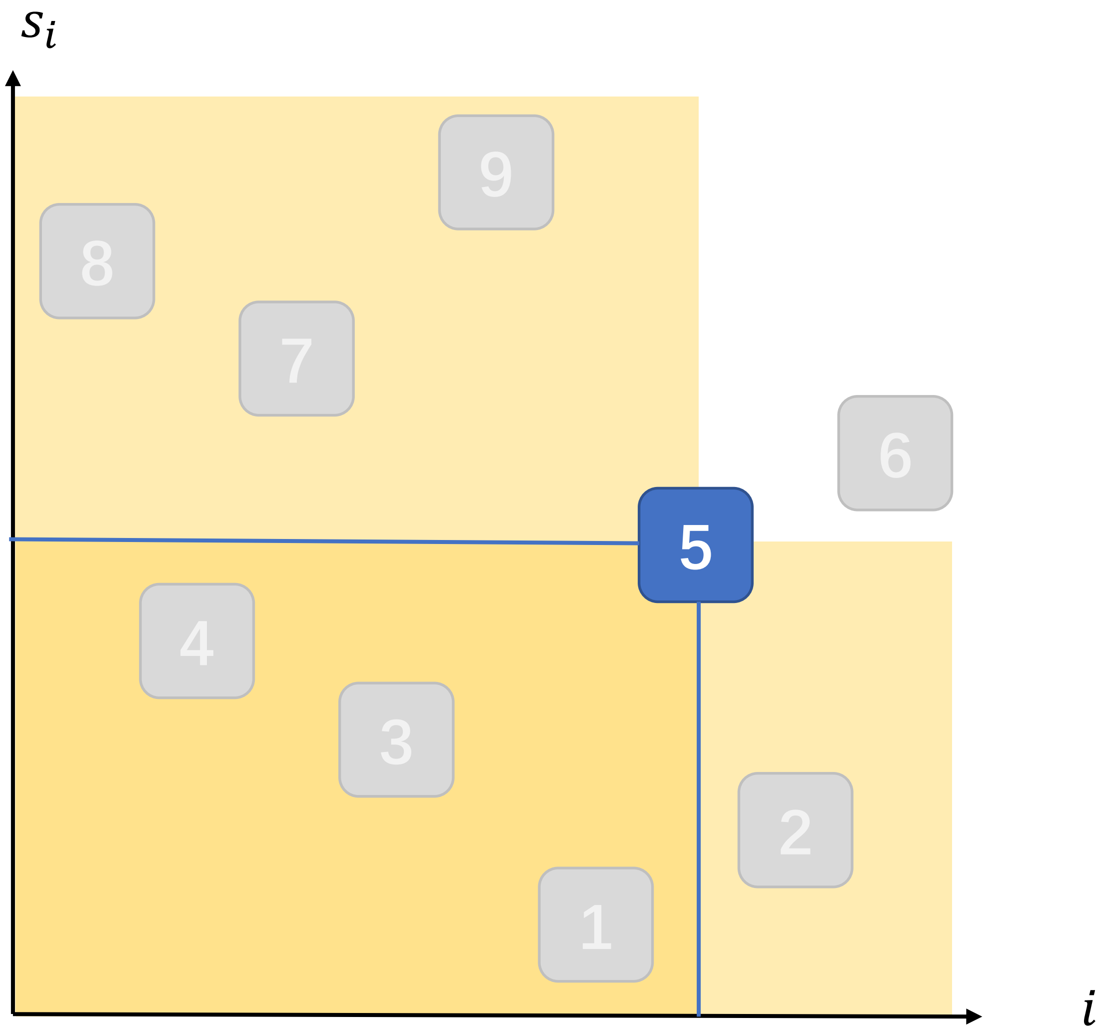

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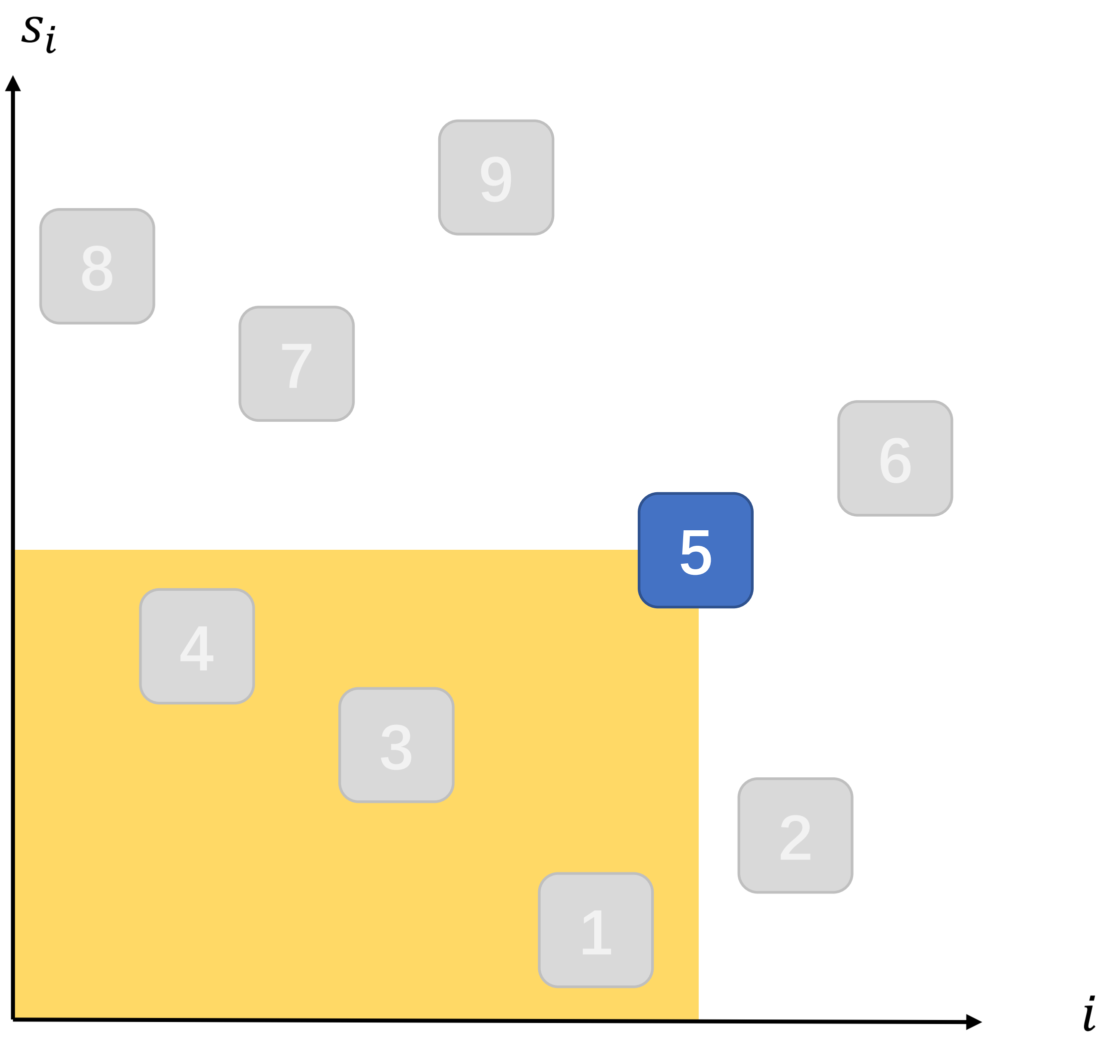
Vertex-centric Approach



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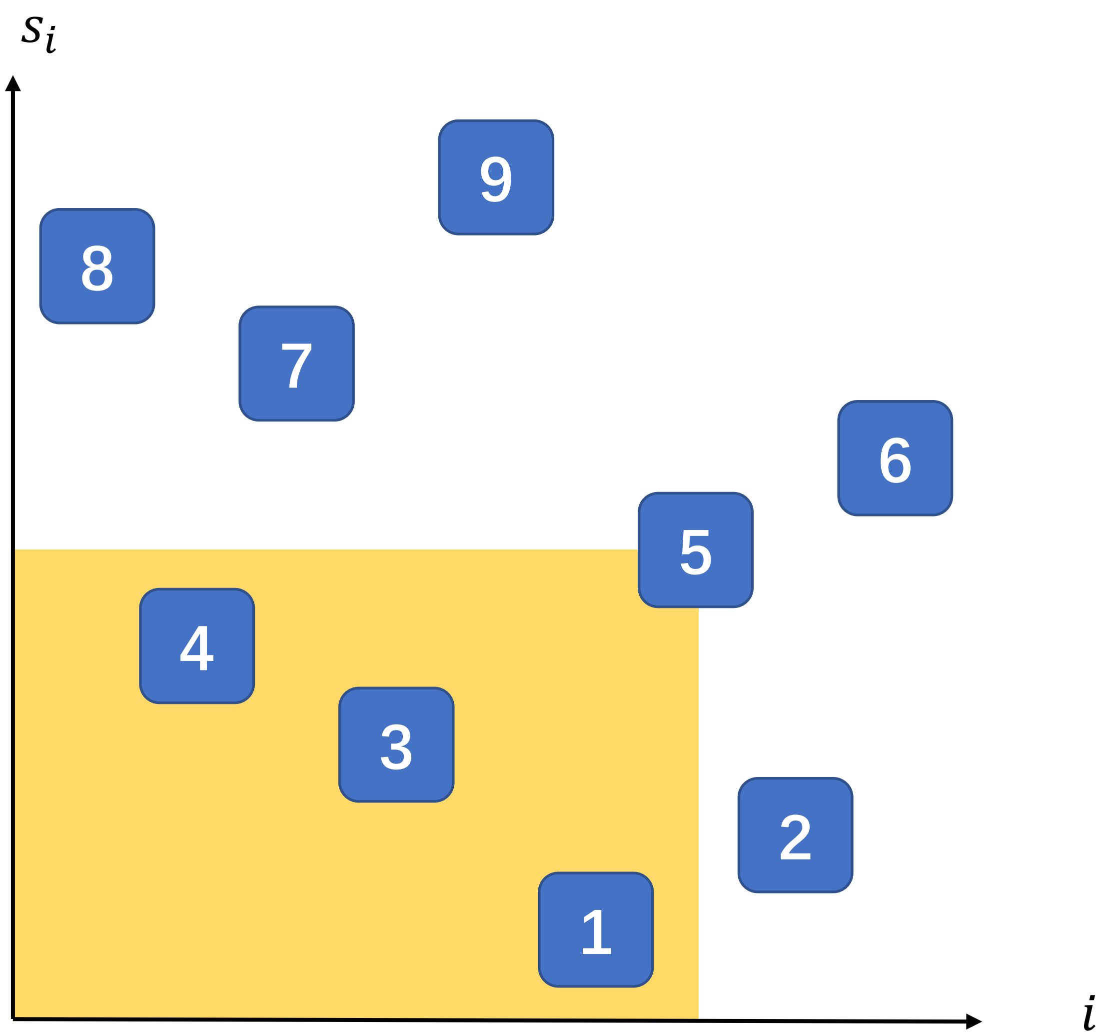

$$dp[i] = \max_{j<i,\, s_j<s_i} dp[j] + 1$$

Readiness check

DP value computation

Pivot re-selection

Above operations can be done by a 2D range tree in $O(\log^2 n)$

[Sun, et al. (PPoPP 2018)]

Complexity of our parallel LIS algorithm

`input`	n
`# wake-ups`	$O(n \log n)$ total wake-ups w.h.p.

`per wake-up`
`- Compute DP values`
`- Check readiness`
`- Re-select pivots`
	$O(\log^2 n)$ by range queries

Parallel LIS Algorithm

Using our phase-parallel framework and vertex-centric methods, we parallelize LIS algorithm with

Nearly work-efficient: $O(n \log^3 n)$ work w.h.p.

Round-efficient: $O(r \log^2 n)$ span

$n =$ input size

$r =$ the LIS length of the input

Key Techniques

Pivots

Wake-up strategy

Vertex-centric readiness check

More applications in **Type-2** approach

- Other range-search-based greedy or dynamic programming
- Maximal independent set (MIS) using wake-up strategy only

	Cost of single readiness check	# candidates in next round
Type-2 approach	Polylog (by range queries)	A few successors (by wake-up strategy)

	Cost of single readiness check	# candidates in next round
Type-1 approach	Trivial	Logarithmic (by 1-D range queries)
Type-2 approach	Polylog (by range queries)	A few successors (by wake-up strategy)

Improved Bounds

* with high probability

	Work	Span
LIS	$O(n \log^3 n)$ *	$O(r \log^2 n)$
MIS	$O(n + m)$	$O(\log^2 n)$ *
Activity Selection (weighted)	$O(n \log n)$	$O(r \log n)$
Activity Selection (unweighted)	$O(n \log n)$	$O(\log n)$ *

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Huffman tree, graph coloring, SSSP, unlimited knapsack, and more in the paper

Experiment Setup

Hardware

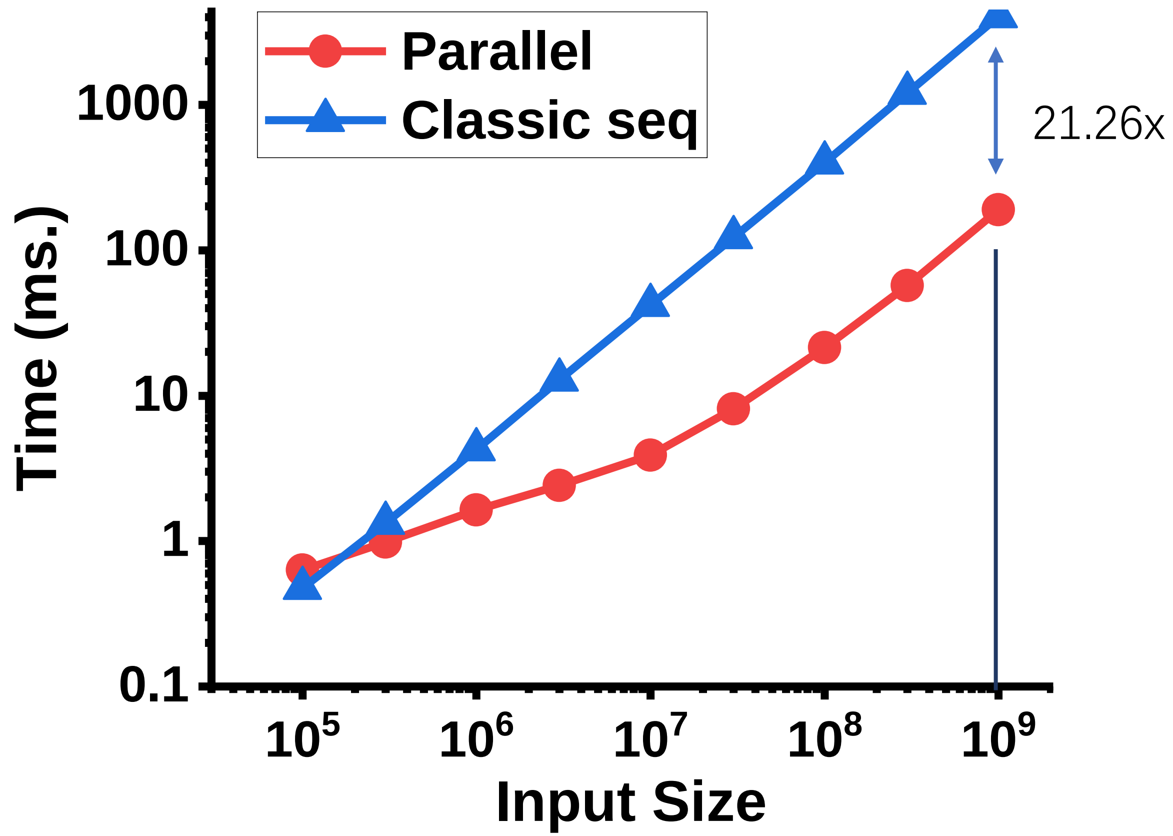
- 96 CPU cores (192 hyper-threads)
- 1.5 TiB of main memory

Parallelized Algorithms

- Huffman Tree
- Activity selection
- LIS
- …

Work-efficient algorithms generally perform well

Huffman Tree



Our algorithm is work-efficient

$\tilde{O}(r)$ span for rank r (tree height)

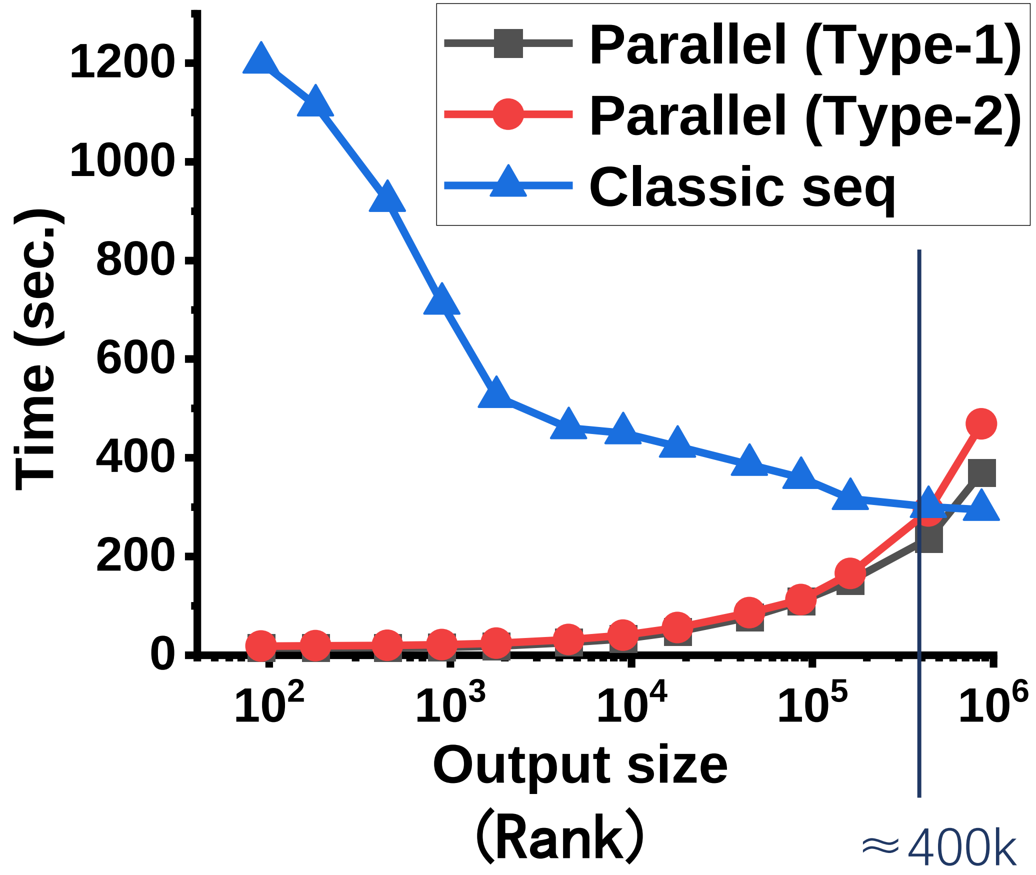
The rank of the test data is low

The algorithm performs well

- 21.26x speedup to the sequential

Activity Selection

$n = 10^9$

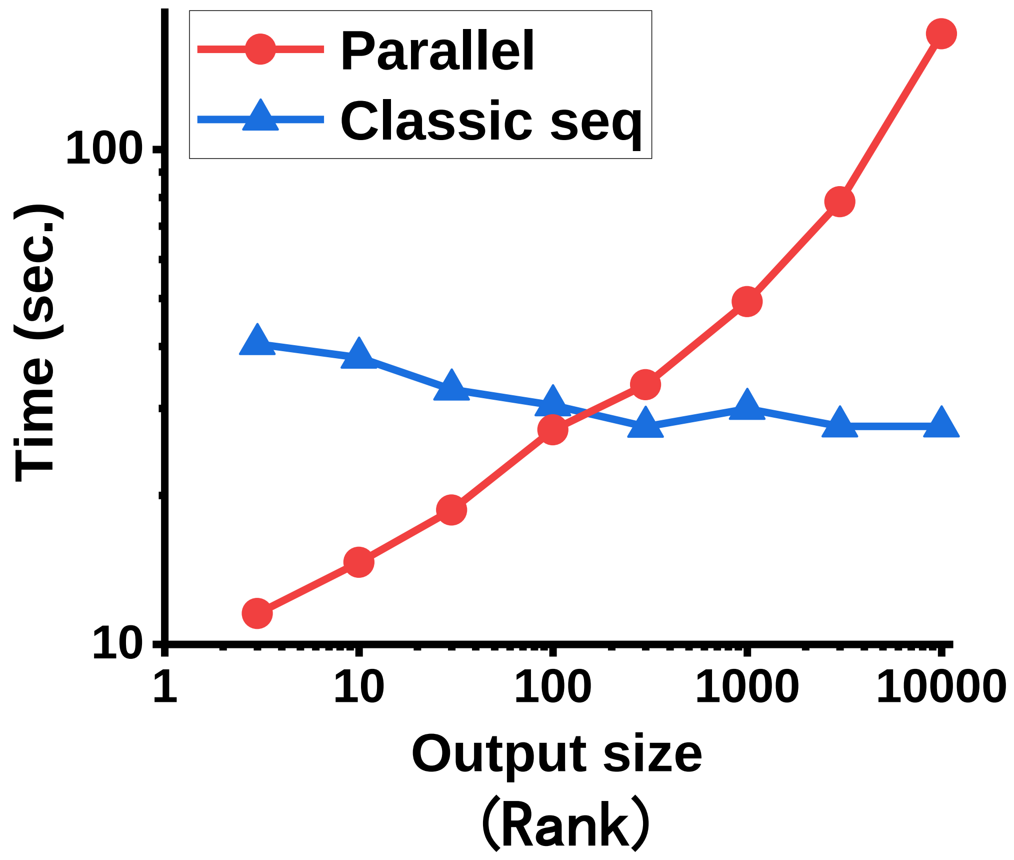


Implement with both Type-1 and -2

- Both are work-efficient: $O(n \log n)$
- Span is $\tilde{O}(r)$ for rank r

The algorithm performs well on a wide range of rank values

Longest Increasing Subsequence



Nearly work-efficient: $O(n \log^3 n)$

- $O(\log^2 n)$ overhead

Run fast on small ranks

- Overhead in work still limits its performance for large ranks

Open problem:

$O(n \log n)$ work with good parallelism?

Summary

Motivation

- Many existing sequential iterative algorithms can be highly parallel
- Parallelize these algorithms efficiently

New Techniques: general to many algorithms

- Phase-parallel framework
- Type-1 and Type-2 approaches

New algorithms (many improving best known bounds)

- LIS, activity selection, MIS, Huffman tree, SSSP, …

Future Work

Can LIS problem be solved in $O(n \log n)$ with good parallelism?

Can our techniques apply to other problems?

Thank you

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